Topological Operators in Quantum Gravity:- A Treatment within Effective Field Theory Consistency

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1 Abstract

Significant progress has been made in recent years on generalized global symmetries . In this paper, we provide a detailed and comprehensive discussion on topological operators , generalized global symmetries , higher -form symmetries with a concrete example using Maxwell's $4D$ theory. We revisit the problem from the perspective of generalized and higher-form symmetries, focusing on the fate of their associated topological operators when they are embedded in gravity. Topological defects have been categorized as solitonic effects in a field profile with appropriate regularization to derive a complete effective action for non-trivial Effective Field Theories(EFTs), which includes gravity. While studying topological operators, it is essential to take into consideration a perturbative regime within which EFTs can be constructed and topological invariance can also be preserved. Considering a d-dimensional EFT that is weakly coupled to Einstein gravity, we continue our investigation into the species scale. A lagrangian that includes the classical gravitation action coupled with matter fields has also been formalised. Taking into consideration the issues that occur with constructing consistent theories of Quantum Gravity in large moduli spaces , within the Planck scale, the paper briefly touches upon Swampland and Swampland Distance Conjecture(SDC)in order to analyse the nature of Goldstone sectors in large field spaces. Finally, we show how missing EFT degrees of freedom can be used to recover topological operators. For both dynamical/non-dynamical charges, we study symmetry-violations and the preservation of topological invariance at larger masses for heavier fields. A suppression factor has also been introduced to mitigate deviations at finite masses.

Contents

2 Introduction

In Classical Field theories, Noether's theorem declares the existence of conserved charges Q, when the action is invariant under a continuous symmetry group.

Now, within the context of global symmetries in Quantum Field theories, significant progess has been made towards developing methods which are Lagrangian independent. This means, we are no longer interested in considering global symmetries, whose development relies on an underlying Lagrangian. Instead, the characterization of global symmetries will depend on the charges. To be more specific, we are going to consider charges and charge objects as abstract operators.

One good example that demonstrates the use of a charge operators can be found in the Witten paper.[1] This paper states that on a general four manifold, there is no natural notion of "time translations," so one must work with a smaller supersymmetry algebra in which the H (The Hamiltonian) does not appear. (there is no globally-defined conserved charge corresponding to time translations.) We have to keep using one charge like generator Q called the BRST operator.

In the context of strings, BRST charge is a crucial operator used for quantization and identifying the physical states $|\psi\rangle$ of the spectrum. [2]
The usage of charges and the charged objects as abstract operators gives us an intrinsic view of the symmetry, which is valid even when there is no Lagrangian.

One natural operation with global symmetries is to couple them to the background gauge fields. This allows us to study "twisted" sectors. If we have a current, we can integrate it over a finite full space-like surface. This partial integration gives rise to the local charges Q , which measure the amount of charge in the region . These are the generators of localized symmetry transformations for local fields in the region. These localized symmetry transformations are called "twists" (holonomies).

The examination of "twists" with respect to coupling with gauge fields for flat backgrounds (that remains gauge invariant) can be done by the insertion of "topological defects". Hence, we can say that this " topological defect" allows us to study the symmetry action on the operators that are represented by it.

Or to put it a little bit more precisely, coupling a global symmetry G to a flat background gauge field can be implemented by inserting a topological defects . These defects act on operators thereby producing a symmetry transform, thus providing a topological probe of the symmetry action and twisted sectors (holonomies).

Quantum field theories are quite good at describing topological operators that generalize the ordinary notion of symmetry in various directions, allowing to transport many familiar concepts (symmetry breaking, anomalies, gauging, etc...) in a much more general framework. TQFT provides us with a powerful framework to live in one higher dimension, which is exactly what is needed when we are considering higher form symmetries.
A global symmetry can be considered as a higher form symmetry when the charged operators are of the space-time dimension q. [3]

A significant amount of work has been done in the last decade to understand different types of operators in QFT. The more general kind of topological operators are those that interplay with charge operators. These generalizations come at different levels of complications.[4] For a symmetry group G whose support is from a higher codimension $q+1$, are called higher form symmetries of degree q . When $q>0$, these symmetries cannot act on local operators , but on operators with dimensionality of atleast q. Therefore, the charged objects here can no longer be considered as particles , but operators that are acting on higher dimensional algebraic structures. Similarly, the charged observables are not zero-dimensional local operators, but higher dimensional objects.

The study of such symmetries is essential to the understanding symmetry actions working on extended operators which were not visible in the old Noether literature. Higher form symmetries can also exist in Effective Field Theories (EFTs) , thus providing a tool in understanding which theories couple better with gravity. We will explore it later in a different section.

These concepts will become more clear as we explore generalized global symmetries . Beginning from an introductory explanation, we are going to work our way up towards topological operators and their behaviour at the Planck scale. This will provide the reader with a clear picture on the approach needed towards understanding topological operators and higher form symmetries and their role in Quantum Gravity.

3 What are Generalized Symmetries ?

3.1 Introduction to Generalized Global Symmetries and Higher Form Symmetry

In Quantum Mechanics, if one has a global symmetry , we know that it acts on states. These operators must be willing to conserve the total probability which is equal to 1. $|\psi\rangle \rightarrow U(t)|\psi\rangle$, where $|\psi\rangle$ is a state and U is the Unitary Operator acting on the state.

Now, if the system evolves in time, then the Unitary operator should also evolve with time. If we want $U(t)$ to represent a symmetry, then it needs to be independent of time. If we have a local operator which changes under this symmetry , we can say that under this time evolution from t to $t^{'}$, the symmetry also changes from $U(t)$ to $U(t^{'})$. [$U(t) = U(t^{'})$.] [5]

$$\mathcal{O}(x,t) \rightarrow U(t)\mathcal{O}(x,t)U^{-1}(t) = \mathcal{O}^{'}(x,t)$$

Here, $\mathcal{O}(x,t)$ is the local operator at spacetime (x,t) which changes according to the evolution in time. If we want to understand the action of the Unitary Operator on the local Operator, we can consider the local operator be at a time t and take two Unitary operators . One at time $t^{'}$ and the other one at time $t^{''}$, such that time t is in between $t^{'}$ and $t^{''}$.

If we try to justify the action on the Unitary Operator $U(t)$, we can try by making $U(t^{''})$ pass through $\mathcal{O}(x,t)$ as seen in Figure 1.(a). This will make the local operator transform from $\mathcal{O}(x,t)$ to $\mathcal{O}^{'}(x,t)$.

Now, it is natural to ask, why could we move the operator from $U(t^{''})$ to $U(t^{'})$. It is because , we saw that $U(t^{''}) = U(t^{'})$ at different times $t^{'}$ and $t^{''}$. From here we can conclude that the scenario does not change unless $U(t^{''})$ passes through the local operator $\mathcal{O}(x,t)$ as seen in Figure 1(b).

So, when a system has properties that doesn't change under continuous deformations, we call such properties as topological properties. The property $U(t^{''}) = U(t^{'})$ can be used to call $U(t)$ as topological operators.

In the previous scenario, we associated a Unitary Operator with a time slice. If the dimension of the time slice is d, then the time slice can be described as $d-1$. Such a surface is called a codimension 1 surface. This means, a surface which is 1 dimension less than the total space. A surface of $d-m$ dimensions will be called a co-dimension m surface.

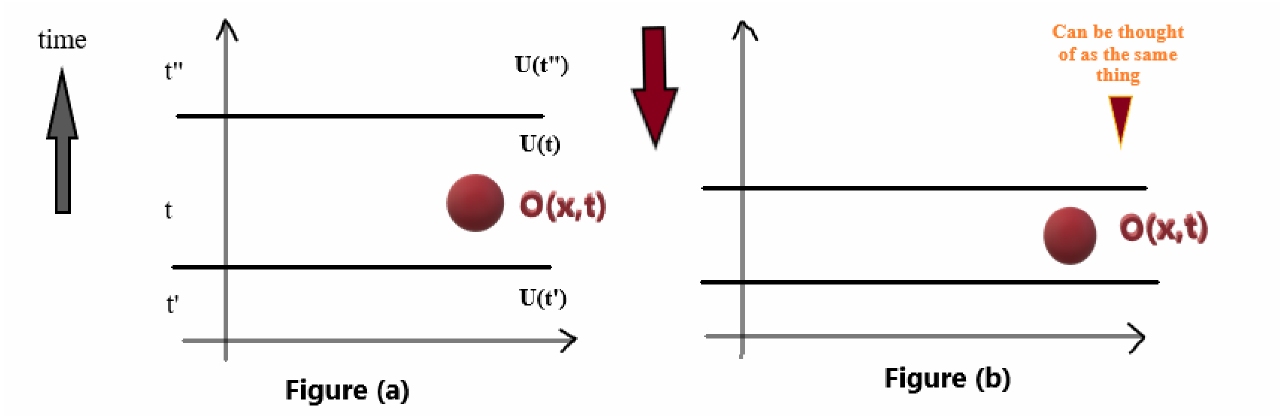

Figure 1: Spacetime is represented here. Different time slices are represented by Horizontal dark lines

$$U(t) \rightarrow U(\Sigma_{d-1})$$

$$U(\Sigma_{d-1}) = U(\Sigma'_{d-1})$$

Here, a Unitary Operator $U(t)$ can be generalized to a codimension 1 surface. Now since, $U(t) = U(t')$. The codimension 1 surface can also transform. In the above scenario with the time slices, when we deformed $U(t'')$, time slices were deformed to other time slices, but now we can also deform a co-dimension 1 surface to any other codimension 1 surface. From the second equation $U(\Sigma_{d-1}) = U(\Sigma'_{d-1})$, we can see the possible generalizations that can be made. Now, the question is , can we do it for other m surfaces ?. It turns out that we can.

The symmetries that we talked about here are ordinary global symmetries whose topological operators, are associated with a codimension 1 surface. As mentioned in the introduction, we can generalize to other form of symmetries with act on operators with a dimension of atleast q called q-form symmetries.

These topological operators are associated with a codimension of $(q + 1)$ surfaces. Their dimension can be written as :- $d - q - 1$.

Before getting into higher form symmetries in details, we have to familiarize ourselves with a bit of a background on topological operators in QFT. In doing so, the journey towards understanding higher form symmetries is going to become easier.

3.2 Topological Operators in QFT

Considering a d - dimensional Eucledian QFT. The theory represents a family of Euclidean Operators (or defects) like for eg:- $\mathcal{O}(x)$. These operators can be labelled by sub-manifolds of various dimensionalities. The theory can be put into an arbitrary Riemannian Manifold represented by X_d, which can produce Eucledian Correlation functions, that can include both local as well as extended operators.

Let us suppose that the theory has a Global Symmetry G with conserved currents represented by $J^a = J^a_\mu \partial x^\mu$ (a is the Lie algebra index, with generators T_a). The currents satisfy the ward identities. [4] [6]

$$\langle d^\star J^a(x)\mathcal{O}_1(x_1)..........\mathcal{O}(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)vol(X_d)\langle\mathcal{O}(x_1).....\delta\mathcal{O}_i(x_j)....\mathcal{O}_n(x_N)\rangle$$

where $\mathcal{O}_i$ is an irreducible representation on $\mathcal{R}_i$. $vol(X_d)$ is the Riemannian Volume form(Volume form of the Hodge dual, which we will see later). Just like we had derived the Ward Identities in the previous paper [6] in the symmetries in QFT section, we are doing for operators on a manifold. The Ward identities can be written in a finite form. Just like in [6] we are considering an open region $D_d \subset X_d$ whose boundary is a compact, homologically trivial submanifold Σ_{d-1} of codimenison 1. Defining the extended codimension 1 charge operatorby integrating over the submanifold. [1]

$$Q^a(\Sigma_{d-1}) = \int_{\Sigma_{d-1}} \star J^a(x)$$

We can derive the modified Ward Identities. The result can be given as:-

$$\langle Q^a(\Sigma_{d-1})\mathcal{O}_1(x_1)..........\mathcal{O}(x_n)\rangle = \sum_{i=1}^{n} L_k(\Sigma_{d-1},x_i)\langle\mathcal{O}(x_1).....\mathcal{R}_i(T_a)\mathcal{O}_i(x_j)....\mathcal{O}_n(x_N)\rangle..... \quad (1)$$

where L_k is the linking Number. $L_k(\Sigma_{d-1}, x)$ is 1 if $x \in D_d$, 0 if $x \in D_d$

We are introducing the Linking Number [2], which remains invariant under any topological deformations of any one of the submanifolds, provided that they do not intersect. Equation(1)

[1] $\star J^a(x)$ denotes the Hodge dual over a $q+1$ manifold. J^a is a differential q form

[2] is a numerical invariant that describes the linking of two closed curves in three-dimensional space. An important concept in knot theory

tells us that the operator $Q^a(\Sigma_{d-1})$ is a topological operator:- correlation functions which are invariant under small deformations. A very crucial property of topological operators is that they are not subject to short distance divergences. In this case, the intrinsic definition of the symmetry can be - Any global group symmetry in QFT, leads to codimension 1 topological operators labelled by $g \in G$.

3.3 Symmetry Operators

When the symmetry is continuous there is a Noether current, which is a high spin current [3]. But these are rarely discussed as Global symmetries. For extended objects i.e, lines, surfaces etc, we will be using the operators. When the spacetime manifold is of the form $\mathcal{M}_{d-1} * \mathbb{R}$ or $\mathcal{M}_{d-1} * S^1$, it is natural to refer to the factor of $\mathbb{R}$ or S^1 as an Eucledian time. In our case, when the observable is placed at a given time, it can be interpreted as an operator acting on a Hilbert space. Also, when the operator is stretched along a time direction, it cannot be considered an operator in theory, but a defect. These kind of global symmetries are independent of the Lagrangian , but the charges and the charged objects are categorized as operators. For ordinary global symmetries $(q = 0)$, we are considering the symmetry transformations to form a group G. If the group is continuous, then we can say that for every continuous generator (that is tied to T^a) , there is a conserved Noether current j which we can write in $d-1$ form.

The conserved charge can therefore be written as:-

$$Q^a(\mathcal{M}^{(d-1)}) = \oint_{\mathcal{M}^{(d-1)}} j$$

where $\mathcal{M}^{(d-1)}$ is a $d-1$ manifold (a codimension 1 surface). Typically, $\mathcal{M}^{(d-1)}$ can be non-compact, unlike in the previous section where the boundary was a compact, homologically trivial submanifold. The symmetry transformation can be written as: - $U_g(\mathcal{M}^{(d-1)})$, where $g \in G$, a group element of the global symmetry.

In the continuous case, the symmetry transformations can be obtained by exponentiating the conserved charge $Q^a(\mathcal{M}^{(d-1)})$. The transformation will satisfy the group law. Here, $g^{''} = g.g^{'}$

$$U_g(\mathcal{M}^{(d-1)})Ug^{'}(\mathcal{M}^{(d-1)}) = U_g^{''}(\mathcal{M}^{(d-1)})$$

$U_g(\mathcal{M}^{(d-1)})$ being associated with a symmetry means that the dependence on $\mathcal{M}^{(d-1)}$ is topological. It remains unchanged when $\mathcal{M}^{(d-1)}$ undergoes slight deformations. This is consistent with what we have encountered in the previous section. It only undergoes deformations when an operator intersects. We will return to this issue in a later section.

We can say that an intrinsic description of global symmetry transformation can be defined in terms of its symmetry operator :- $U_g(\mathcal{M}^{(d-1)})$

3.3.1 Higher-Form Symmetry

In d spacetime dimensions, a q- form global symmetry transformation is applied by a symmetry operator $U_g(\mathcal{M}^{(d-q-1)})$, which has a support on a $q+1$ closed manifold $\mathcal{M}^{(d-q-1)}$ [3]. In the previous section we have mentioned that the operator can only undergo small deformations when another operator which is charged under the symmetry, crosses it.

The charged object of a q- form global symmetry is an operator $V(\mathcal{C}^{(q)})$, which is supported on a q- dimensional closed manifold $\mathcal{C}^{(q)}$ in spacetime.[7] The action of the symmetry on the operator V is given as [4]:-

$$U_g(\mathcal{S}^{(d-q-1)}).V(\mathcal{C}^{(q)}) = g(V).V(\mathcal{C}^{(q)})$$

where $g(V)$ is the representation of g realized by the charge operator V.
When $G = U(1)$, there will be a conserved $(q+1)$ - form Noether Current j. In that case, the symmetry Operator would simply be:- $U_\theta(\mathcal{M}^{(d-q-1)}) = exp[i\theta \int_{\mathcal{M}^{(d-q-1)}} \star j$, where $\theta \in [0, 2\pi)$ for the $U(1)$ group.

3.4 Noether Currents and Global Symmetries

So far, we got a preliminary idea of what topological operators, generalized global symmetries and higher form symmetries are and what they entail. We are going to bring it together by using a concrete example. We know that when $G = U(1)$, there will be a conserved charge j. Putting it into perspective with global symmetries, we are going to derive some intuition on what exactly prevents generalized global symmetries from getting gauged away.

Consider a symmetry group G of a field theory which leaves the action invariant. In that case, Noether's theorem proves the continuity equation $d^\star J = 0$. As we have seen in the

[3] q-form symmetry in d dimensions is implemented by an operator associated with a codimension $q+1$ closed manifold
[4] We are linking a sphere S^{d-1} with $U_g(\mathcal{S}^{(d-q-1)})$

previous section, a conserved charge can be derived because there is a conserved Noether current. The charge can be obtained by integrating over the conserved Noether current over a codimension 1 manifold:-

$$Q^a(\mathcal{M}^{(d-1)}) = \oint_{\mathcal{M}^{(d-1)}} j$$

Now, in a Quantum Field Theory setting, we can insert the current into correlation functions and derive the Ward identities as seen in section (3.2). For example:- a simple local operator $\mathcal{O}$ inserted at a point p, this will be:-

$$\delta^{(d)}(x-p)\langle\delta\mathcal{O}(p)\rangle = \partial_\mu\langle J^\mu(x)\mathcal{O}(p)\rangle$$

Integrating this equation over a d- dimensional Riemannian volume X_d, it will give us:-

$$\langle Q^a(\mathcal{M}^{(d-1)}\mathcal{O}(p)\rangle = L_k(\mathcal{M}^{(d-1)}, p)\langle\delta\mathcal{O}(p)\rangle \tag{2}$$

where $L_k(\mathcal{M}^{(d-1)}, p)$ is a topological property . The charge is independent of the manifold $\mathcal{M}^{(d-1)}$, as long as the Linking number is non-trivial with p. (does not undergo any deformations). [8]

For a continuous symmetry $G^{(0)}$ [See Appendix - 10.1.I] , a $U(1)$ symmetry with a conserved current , we can define a topological operator that implements a symmetry action on the operators.

In this case, we can define an operator $U_\theta(\mathcal{M}^{(d-1)})$, which implements the action of the symmetry on the operator. The representation θ can be parameterized as: - $\theta = e^{i\alpha} \in U(1)$. [5].

$$U_\theta(\mathcal{M}^{(d-1)}) = exp[i\theta \int_{\mathcal{M}^{(d-1)}} \star j]$$

We can insert this in the correlators of equation(2) and find that for $U(1)$, the topological operator acts by a phase. $U_\theta(\mathcal{M}^{(d-1)})$ - This operator is topological as the associated current is conserved and thus any small deformation in the manifold $\mathcal{M}^{(d-1)}$ is immaterial , as long as the local operator is inserted not crossed. Proof is shown at the end of this section. See figure (2).

Example:-

[5] α is a group element for non invertible symmetries

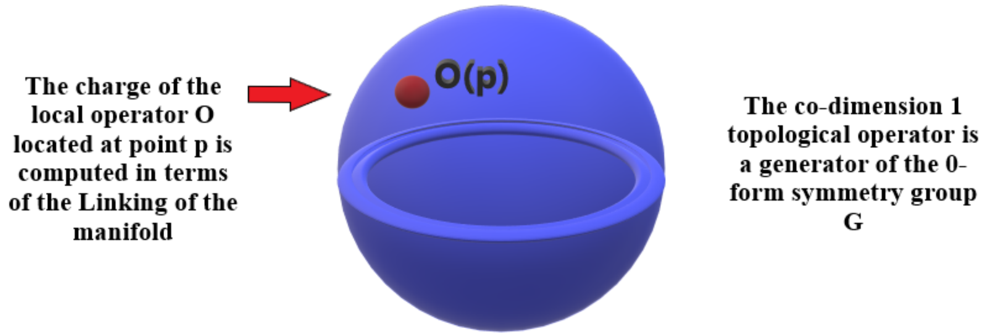

Figure 2: The charge is invariant under the deformations of the manifold $\mathcal{M}^{(d-1)}$, which do not change the linking as long as the operator insertion is contained within the volume that the manifold bounds. $\theta \in [0, 2\pi)$

- For a $U(1)$ gauge theory on $4D$, we consider the action of charge conjugation , which maps the gauge field, $A \to -A$, where $\theta \in [0, 2\pi]$ for the $U(1)$ group. This indicates that the field strength $F(x)$ is acted upon by $F(x) \to -F(x)$, thus a non-trivial 0 -charge. ($q = 0$, ordinary symmetries) [9]

- For a 1- form symmetry, where $q = 1$, $U_2^{(\theta,i)} = U_2^{(-\theta,i)}$.

where i is equal to Electric Lines or Magnetic Lines and acts on the Wilson Line operators. $W_\alpha = \mathcal{P}exp[i\int A]$, where $\mathcal{P}$ is the path ordering operator. [6]

Proof:-

Under a manifold $\mathcal{M}^{(d-1)}$, the charge can be written as :- $Q(\Sigma_{(d-1)}) = \int_{\Sigma_{(d-1)}} \star J^a$. Upon any minor deformation $Q(\Sigma'_{(d-1)})$, the charge remains invariant.

$$Q(\Sigma_{(d-1)}) - Q(\Sigma'_{(d-1)}) = \int_V d \star J = 0$$

Conservation law here implies that transformation U depends on the topology.

4 Higher Form Global Symmetries

4.1 Higher Form Symmetry in $4D$ Maxwell's theory

Maxwell's theory can be considered as an example of a physical theory with higher -form symmetries. The action of the theory can be defined in terms of a compact $U(1)$ gauge field A_μ. The action can therefore be defined as:-

$$S = \int -\frac{1}{2e} f \wedge \star f = \int d^D x - \frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} \tag{3}$$

where $f_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu$ from the Bianchi identity and the coupling constant $[e^2] = 4 - D$.

Now according to equation (3), what should be our observable. The answer lies in the theory being related to gauge -invariant objects, which can be constructed from the electromagnetic field strength $f_{\mu\nu}$. These can be considered as local operators. [10]We can construct extended operators which are gauge invariant . They are the Wilson-line, defect operators. It can be written as:-

$$W_{q_w}[\mathcal{C}] = exp[iq_w \oint_{\mathcal{C}} A_\mu] \tag{4}$$

where $\mathcal{C}$ as mentioned in the previous section, represents a closed manifold . This is necessary, in order to be gauge invariant. The parametre $q_w \in \mathbb{Z}$ is the charge of the Wilson line. [6]

Now, question is- what is the physical interpretation from the Wilson line operator in the equation(4) ?. We can say that it represents the worldline of a probed charged particle, this means a particle which has no dynamics. The expectation value here, can therefore be written as:-

$$\langle W_{q_w}[\mathcal{C}] \rangle = \int DA_\mu exp[iq_w \oint A_\mu] e^{is[A_\mu]} \tag{5}$$

We can consider a conserved current that is moving along a curve represented by- $\overrightarrow{y}(x^0)$. Therefore, the Wilson line can be written as:-

$$W_{q_w}[\mathcal{C}] = exp(i \int d^D x J^\mu A_\mu) \tag{6}$$

Now, the expectation value of the Wilson loop can correspond simply to coupling the theory with a non-dynamical charged matter, which is parameterized by the current J^μ .

The expectation value can therefore be written by combining equations (5) and (6).:-

$$\langle W_{q_w}[\mathcal{C}] \rangle = \int DA_\mu e^{iS[A_\mu] + i \int d^D x J^\mu A_\mu}$$

The equations of motion can therefore be derived from equation(3) and can therefore be written as:-

[6]The Noether charge is not to be confused with Wilson line charge. The Noether charge is a conserved quantity, meanwhile, the Wilson line charge is a gauge invariant observable. It acts as a charged operator

$$\frac{1}{e^2} d \star f = 0$$

and

$$df = d \star (\star f) = 0 \tag{7}$$

which in component terms can be written as:-

$$\frac{1}{e^2} \partial_\mu f^{\mu\nu} = 0$$

and

$$\partial_{\mu_1} \star f^{\mu_1 \mu_2 \ldots \mu_{D-2}} = 0 \tag{8}$$

Now, equations (7) and (8) together imply, two higher form symmetries. Namely, a electric 1-form symmetry and a $(D-3)$ form magnetic symmetry with currents $J_e = \frac{1}{e^2} f$ and $J_m = \frac{1}{2\pi} \star f$. The corresponding charges will therefore be:-

$$Q_e(\Sigma_2) = \int \Sigma_{(D-2)} \star J_e = \frac{1}{e^2} \int \Sigma_{(D-2)} \star f$$

and

$$Q_m(\Sigma_2) = \int_{\Sigma_2} \star J_m = \frac{1}{2\pi} \int_{\Sigma_2} \star(\star f) = \frac{1}{2\pi} \int_{\Sigma_2} f$$

where $U(1)_e^1 * U(1)_m^{(D-3)}$ are the symmetries.

In $D = 3$, the magnetic symmetry is an ordinary 0- form symmetry and for $D = 4$, both electric and magnetic are 1- form symmetry. 't Hooft operators are the natural candidates to be the charged objects under $(D-3)$ form symmetry. We can also prove that the action S can remain invariant under gauge transformations. A detailed treatment of 't Hooft anomalies has been done in section (4.2).

4.1.1 Action invariance

For Abelian Gauge theories, we are considering the simplest pure $U(1)$ gauge theory, namely Maxwell's theory. Given as:-

$$S = \frac{1}{4e^2} \int_{X_d} F \wedge^\star F$$

$F = dA$ is considered as the curvature of a $U(1)$ connection A , and it satisfies the Dirac condition $\int_{\Sigma_2} \frac{F}{2\pi} \in \mathbb{Z}$, with $\Sigma_2 \subset X_d$ (any compact 2-manifold). The path integral sums over all the topological classes of $U(1)$ bundles, meaning that all possible integer value of the fluxes are included and for each given bundle, we sum over all of the possible connections A(gauge transformations).

$$A \to A^{\lambda} = A + d\lambda$$

λ is a globally defined closed 1- form with quantized periods. Now taking into considering a loop γ_1 over the holonomy representation. $\int_{\gamma_1} \lambda = 2\pi n\gamma_1$, $n\gamma_1 \in \mathbb{Z}$. For $n\gamma_1 = 0$ on all of γ_1, gauge transformation λ is small. Otherwise it is large. [4] Both these cases are examples of redundancies of the path integral. Hence, no operator can transform non-trivially under them and they do not correspond to any type of global symmetry . This implies that the Wilson line must have integer charge.

$$W_n[\gamma_1] = e^{in} \int_{\gamma_1} A$$

The main observation here is that shifting the connection $A \to A + \xi$ by a closed 1 - form ξ with non-quantized periods, will still leave the action invariant . But, the Wilson lines can transform non-trivially. [To obtain gauge-invariant observables, Wilson lines are often coupled with charged fields , where their non-trivial transformation is canceled by the transformation of the associated field].

$$A \to A + \xi \Rightarrow W_n \to e^{i\alpha} W_n(\gamma_1)$$

where

$$\alpha = \int_{\gamma_1} \xi \in \mathbb{R}/2\pi\mathbb{Z} \simeq U(1)$$

hence, this is not a redundancy, but a true global symmetry of 1- form , which is exactly what we are looking for [it is unlike λ, where gauge transformations are small]. Here, $\alpha = \alpha + 2\pi$, because 2π periods belong to large gauge transformations.

4.2 Anomalies and Inflow

We know that in Quantum Gravity, symmetries are either broken or gauged away. One operation that we can perform with global symmetries, is to couple them to the background gauge fields. If the system has a global symmetry G, we can introduce a principle G bundle with a connection A, which is coupled with QFT. For a continuous symmetry, we have conserved currents J^a, and a coupling is achieved by adding a term in the action (as we have seen in the section (4.1) by combining equations (5) and (6)).

$$S[A] = S + i\int_{X_d} A_a \wedge \star J^a$$

If we consider the $U(1)$ case for simplicity, the gauge field is subject to gauge invariance where $A \to A + d\lambda$. For flat background coupling , $A \wedge \star J$, can be interpreted as the insertion of the topological defect configuration. So, the coupling with the current is equivalent to the insertion of:-

$$U_\alpha(\Sigma_{(d-1)}) = exp(i\alpha \int_{\Sigma_{(d-1)}} \star J^a)$$

in the path integral with α determined by the Holonomy of the gauge field. A gauge transformation $A \to A + d\lambda$, does not change the correlation function because of the topological operator. Equivalently, gauge invariance is a consequence of the conservation $d \star J = 0$.

There are interesting situations where this conservation equation holds in the absence of gauge fields, it breaks as soon as the symmetry is coupled to the background. Hence, if the theory has a symmetry, it is impossible to couple it to a background field that preserves the invariance of the partition function under background gauge transformation. Hence, the theory can be said to have 't Hooft anomalies.[11] Let's consider an example:-

If a current J^a is coupled to a background field A, the conservation equation will be violated by an A dependent form.[4] This means that the partition function coupled with A is not gauge invariant under $A \to A^\lambda = A + d\lambda$, but it picks up a phase. [See Appendix - 10.2.II]

This can be corrected by modifying the partition function by introducing a local anomaly functional L of the background field namely, $S = \int_{X_4} L(A)$. Now, if we can assume that the bulk where X_4 is the boundary of X_5 $5d$ manifold where A extends to the bulk [7] (Chern-Simons), we can write the equation as:

$$S_{inflow} = i\int_{X_5} A \wedge dA$$

The gauge variation of this functional will cancel out the anomalous variation (which is the phase). This mechanism is called anomaly inflow. [12] [13]

[7] Chern-simons action for a background $U(1)$ gauge field , global symmetry in $4d$ described by a $5d$ chern-simons action

Now, one can classify different types of 't Hooft anomalies under the assumption that the anomaly can be cancelled by an anomaly inflow from a classical topological field theory in one higher dimension. If this is the case, an anomaly can be characterized as the action of this theory. For eg:- consider a theory in d- dimensions that has a $U(1)$ global q- form symmetry. Then, possible 't Hooft anomalies correspond to topological actions in the $(d+1)$ dimenisons built out of a $(q+1)$ - form gauge field $\mathcal{C}$ (closed manifold). These actions should be viewed as classical . They might not lead to possible quantum theories.

So far, our discussion has been limited to the exploration of generalized global symmetry and higher- form symmetries. We have used a physical theory like $4D$ Maxwell's theory to dive deep into gauge invariant operators like Wilson and 't Hooft loops. 't Hooft anomalies and anomaly inflow has also been discussed in depth. In the next section, we are going to explore what happens to topological operators when they are embedded in gravity, which is the primary motivation of the paper.

5 Topological Operators in Quantum Gravity

So far, we have understood that ordinary global symmetries can be implemented by co-dimension 1 topological operators, where the symmetry transformation can be given as $U_g(\mathcal{M}^{(d-1)})$. These can be considered as symmetry operators , which are a special case of topological operators. These operators are implemented by a group element g of the global symmetry.

In $4D$ Euclidean space, if we have an operator which has some charge and we wish to measure such a charge, on a three -dimensional surface surrounded by four dimensions, we can shrink it and obtain the charge, because the charge remains invariant under any deformations of the manifold $\mathcal{M}^{(d-1)}$. We have shown this in figure (2).

Now, assume we have two of these operators. We can stack them and perform multiplication (product) on these two symmetry operators. See figure(3.a). In pure QFT, these stackings defines a fusion category of symmetry operators.

So why is this topological ?

The fusion property of topolgical operators refers to the fusion rule which describes how topological excitations combine under TQFT or topological order. [4]. These rules describe an algebraic structure , dictating what results combine them. The algebraic structure is fundamental for understanding the properties of such operators. As mentioned previously in section(3.2) , a crucial property of such topological operators is that they are not subject to short distance divergences(non-triviality). Therefore, there is a well-defined notion of stacking two or more operators and fusing them . See Figure (3.b). Hence, this allows us to perform products among them. From the Ward identities in eq(1), it follows that:-

$$U_g(\Sigma_{(d-1)}).U_h(\Sigma_{(d-1)}) = U_{gh}(\Sigma_{(d-1)})$$

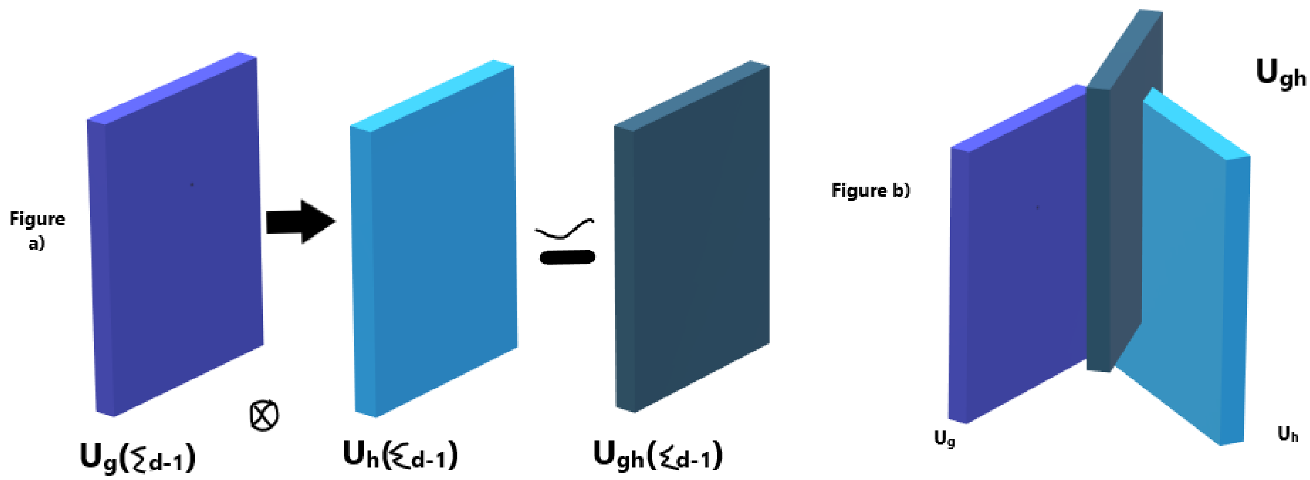

Figure 3: Figure 3.a)-On the Left, Parallel fusion between two topological operators g and h. It produces an operator gh. On the right Figure (3.b) two topological defects meet and a third one is obtained via fusion

The general fusion group of topological operators is NOT group-like. The above equation implies non-invertible symmetry generators. The usefulness of the fusion property of topological operators can be defined at lower energies.[14] At low energy, the fusion proprty characterizes the algebra and the properties of topological operators. There are no topological sectors at high energies.

5.1 Operator insertions

An important question that needs to be examined here from a QFT perspective is - What happens when we insert an operator in field theory ?.

If we have fields in our system, across the operator , the field transforms to whatever value we want it to transfer to. So, considering a $U(1)$ symmetry, the field would also have some $U(1)$ transformation. Upon insertion of an operator, the field would have some step-function jump, which we are identifying using Φ and Φ', where Φ' is the rotation under the symmetry transformation.

Due to issues relating to uncontrolled divergences, the need for renormalization and an overall difficulty in constructing physical theories, QFT encounters issues relating to singular insertions. Typically for topological operators, we exponentiate the current and we insert it into a path integral and upon expanding it, we encounter certain contact terms. A simple operator insertion isn't going to be good enough , thus there's a need for regulation. By giving the field a width of say λ, we can regulate it in some way.

We can implement a boundary condition far from the width. This perspective defines a topological operator as a soliton with finite width and the field profile [8]as a domain wall. If we wish to play with it within some correlation function, we should treat it like its a soliton. [9]The profile $f_\lambda(t)$ is a regulator for this insertion for the topological operator. The field profile $\Phi f_\lambda(-t) + \Phi' f\lambda(t)$ should yield the topological insertion in $\lambda \to 0$, upto counterterms.

There are collective coordinates that are associated to the insertion. These are related to the Goldstone modes which emerge from spontaneous symmetry breaking of the translational symmetry of the field. Another way to think about this is that when we make an insertion, we have wiggles. These wiggles can be considered as Goldstone modes.

The regulator allows us to study the dynamics of the collective coordinates $t(x)$. The width λ is related to the coupling of $t(x)$. In the limit of $\lambda \to 0$, the collective coordinate must either freeze or decouple in order to recover the topological operator.

This idea can be explored using an example.

5.2 Scalar field with Shift Symmetry

For this context, let's consider a scalar field with Shift Symmetry.[14] The action of a free scalar field $\phi(x)$ in a d- dimensional flat spacetime can be denoted as:-

$$S = \frac{1}{2} \int d^d x \partial_\mu \phi \partial^\mu \phi \tag{9}$$

This equation is invariant under a shift transformation $\phi \to \phi + \alpha$, where the associated Noether current is :- $j_\mu = \alpha \partial_\mu \phi$. Upon transformation , the field configuration can be written as:-

[8] Just inserting an operator and transforming a field doesn't provide a complete definition . We have to consider it a field profile for understanding the full picture

[9] Wave-like packet whose stability arises from topological constraints. Here, we are treating the operator insertion as a field profile, where we have some asymptotic boundary condition and we are treating it as a soliton.

$$\phi \to \phi + \alpha f_\lambda(t - t(x)) \tag{10}$$

where $\lim_{\lambda \to 0} f(\lambda) = \Theta(t)$, Θ is the Heavy side step function. The solitonic object can be plugged into the action. Upon expansion, the action can be denoted as:-

$$S = \int d^dxdt[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \alpha f'_\lambda(\partial_t\phi - \partial_\alpha\phi\partial^\alpha t(x))] + \frac{\alpha^2}{2}(f'_\lambda)^2(1 + (\partial_\alpha t(x))^2)$$

The λ going to a zero limit, f goes to a Θ function. It's derivative goes to a delta function.

$$S = \int d^dxdt[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi] + \int d^dx[\alpha(\partial_t\phi - \partial_\alpha\phi\partial^\alpha t(x)) + \frac{\alpha^2}{2\lambda}(1 + (\partial_\alpha t(x)^2))] \tag{11}$$

The term $(\partial_t\phi - \partial_\alpha\phi\partial^\alpha t(x))$ recovers the Noether current associated with the shift. Hence, we can say that the insertion of a topological operator and the treatment of the insertion as a field profile does indeed insert the current the way that we expected. This is the Noether current of Shift Symmetry.

There's a term which appears in the above equation (11) $\partial_\alpha t(x)^2$, which is the dynamics of the collective coordinates. These are the dynamics of the Goldstone modes. Universally, we are going to get some operator which is denoted by 1 in the above equation. This is something which we can remove by some counterterm. This is the energy of the defect.

In the equation(11) if $\lambda \to 0$, the dynamics of a $t(x)$ and the Goldstone decouples. The current term survives and yields the insertion of the topological defect. As the λ term goes to zero, we can think about it as a low- energy limits of the dynamics. Freezing the Goldstone mode is the only way to recover the operator.
This example demonstrates that regulated operator insertions recover the expected topological defect operator in the zero-width limit, while intermediate $\lambda > 0$ encodes defect dynamics through Goldstone fluctuations. This formalizes the intuitive statement that topological operators in QFT can be understood as the $\lambda \to 0$ limit of regulated solitonic profiles.

5.3 Topological defects

So far, we have argued that topological operators that are expressed in terms of currents need regularization. We can accomplish this by giving the operator a finite width. A regulated topological defect $U(\Sigma)$ will not only acquire a non-zero width, but it will also fluctuate. These fluctuations will freeze in a zero-width limit and we can thus recover the operator. Essentially, we are looking for the effective action (S_G) for the Goldstone mode upon introducing a defect.[14] Let's demonstrate this using an example.

To begin, let Σ be a co-dimenison 1 surface at $t = t_0$ for a solitonic model using similar field profile configuration as the previous section. We can consider a field which constantly interpolates between $+\infty$ and $-\infty$. In a bulk Quantum theory , a finite width object fluctuates around its classical position $t = t_0$. We can model these fluctuations by promoting $t \to t_0(\sigma^\alpha)$ to a collective coordinate which depends on the worldvolume coordinates of a defect. This mode can be interpreted as the Goldstone mode for a broken symmetry. The effective action should comprise of both universal and non-universal terms, in which the latter depend on the specifics of the defect at hand. The diffeomorphism invariant universal term that contains the most relevant kinetic terms is the Nambu-Goto action of the worldsheet [10].

$$S_{NG} = \frac{T}{2}\int_\Sigma d^{d-1}\sigma\sqrt{det\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}}$$

where T is the base tension. The defect's classical position is the flat plane at $t = 0$. Expanding about small fluctuations, the defective action becomes.

$$S_{NG} = \frac{T}{2}\int_\Sigma d^{d-1}\sigma(1 + (d_\alpha t_0)^2 + ...)$$

Notice the similarity in the defective action with equation(11) in the previous section. From the previous section, if we substitute equation(10) into equation(9), we get equation(11) upon expansion. When the value of λ in equation (11) is taken to be sufficiently small, the overall action can be approximated as:- $S = S_{bulk} + S_d + S_G$, where S_d is the regulated defect and S_G as mentioned earlier, is the effective action for the Goldstone mode.

$$S_G = \int_\Sigma d^{d-1}\sigma[\frac{T_{eff}}{2}((d_\alpha t_0)^2 - \alpha\partial^\alpha\phi\partial_\alpha t_0)]$$

Just like in equation(11) , the term $(d_\alpha t_0)^2 - \alpha\partial^\alpha\phi\partial_\alpha t_0$ represents our shift which we are looking for. In the above equation, $T_{eff} = T + \frac{\alpha^2}{6\lambda}$

For $\lambda \to 0$, the collective coordinate t_0 freezes to become a constant. Hence, we can say that the zero-width limit provides a regulated version of the topological operator. [The infinite constant can be absorbed in the Normalization factor].

[10] Denotes the worldsheet co-ordinates of the defect. This is analogous to the string worldsheet, where the action of the worldsheet is given by the Nambu-Goto action

5.3.1 Higher-Form Symmetries

The freezing of Goldstone modes can also be proved by using Higher form symmetries. Just like in Section(4.1), we have to use a physical theory like Maxwell's theory. For a q -form symmetry, the charged object can be represented by $V(C^{(q)})$ where $\mathcal{C}^{(q)}$ is a closed manifold.

To probe a $U(1)$ field , 1- form symmetry, it is convenient to use a backround gauge field given by A_μ. The conservation equation $d \star J = 0$ can be understood by a shift in the gauge field :- $A_\mu \to A\mu + \xi$ where ξ is a closed 1- form . The action given by equation(3) in section(4.1) has to remain invariant under A_μ , a properly normalized flat $U(1)$ gauge field. (See Appendix 5.2.II). [15]

The theory has a Wilson line $U(\mathcal{M}^{(1)}) = exp(i \oint_{M^{(1)}} A_\mu)$.

Just as described in the previous section , we can use a solitonic model, field profile configuration where F is the field strength and F' is the field strength away from the symmetry defect. Once we regulate the field profile and introduce the Goldstone modes, the regulating functions can be:- $f_\lambda(t^i - t_0^i)$. Hence the Goldstone modes will be t_0^i.

By using the solitonic model , we can start with a co-dimension 2 sub-manifold Σ_2. The step function can be described as:- $\Delta t^i - t_0^i$.

Once again, just like the previous section the over action can be approximated as -
$S = S_{bulk} + S_d + S_G$.

$$S_G = \int d^2\sigma[\frac{T}{2}(\partial^\alpha t_0^i)^2 + \alpha\partial_\alpha t_{0,i} j^{\alpha,i} + \frac{\alpha^2}{72e^2\lambda^2}(\partial_\alpha t_0^i)^2 + ...]$$

The terms that are indicated by the dots in S_G include terms of higher order in the Goldstone modes that reproduce the Nambu-Goto action as well as higher order couplings to the bulk field strength; the latter are independent of λ. In the zero-width limit, the Goldstone modes will freeze out.

5.4 Topological Operators in the Presence of Gravity

In this discussion so far, we have modeled the symmetry defect as solitonic model regulated under a field profile configuration. Generally speaking, the approximated effective action on a codimension $(q+1)$ defect is going to be:-

$$S_{NG} = \frac{T}{2}\int_{\Sigma} d^{d-q-1}\sigma\sqrt{det\frac{\partial x^{\mu}}{\partial \sigma^{a}}\frac{\partial x^{\nu}}{\partial \sigma^{b}}} + ...$$

The collective coordinates are denoted by:- $x^i = t^i - t_0^i$. [x^μ denotes $x^\alpha = \sigma^\alpha$].

We would like to consider the bulk theory coupled to gravity . The gravitational metric would take the form :- $g_{\mu\nu} = \delta_{\mu\nu} + \mathcal{K}h_{\mu\nu}$, where $h_{\mu\nu}$ is the graviton field and the perturbative parametre is $\mathcal{K} = 32\pi^2 G_N$. , G_N is Newton's constant. Just like in the previous sections, expanding upon smaller Fluctuations, the effective action would therefore become:-

$$S = \int d^{(d-q-1)}\sigma\frac{\mathcal{K}.T}{2}(h_{ij}\partial^{\alpha}t_0^i\partial_{\alpha}t_0^j + +h_{\alpha b}\partial^{\alpha}t_0^i\partial^b t_{0,i} - 2h_{\alpha i}\partial^a t_0^i + h_{\alpha}^{\alpha}) + ...$$

If a we consider a field with an induced metric g, the Lagrangian of the field can be given by:-

$$\mathcal{L} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi$$

For the Lagrangian to remain invariant under a shift transformation $\phi \to \phi + \alpha$ exactly as in section(5.2), the total action of the theory can be written as:-
$S = S_{bulk} + S_{op}^{\lambda}(\phi, t) + S_{graviton}^{\lambda}(\phi, t, \mathcal{K})$, where $S_{op}^{\lambda}(\phi, t)$ is the expected term for the field theory. We also get a new term for the graviton which is $S_{graviton}^{\lambda}(\phi, t, \mathcal{K})$. By doing this, we will receive some non-trivial EFT which couple the graviton with the Goldstone mode. The graviton action can therefore be given by:-

$$S_{graviton}^{\lambda} = \int d^3x[\frac{\alpha^2\mathcal{K}}{12\lambda}(h_{tt} + h_{\alpha b}\partial^{\alpha}\partial^b t - 2h_{\alpha t}\partial^{\alpha}t)] + \mathcal{K}\alpha[h_{tt}\partial^t\phi - h_{\alpha b}\partial^{\alpha}\phi\partial^b t + h_{\alpha t}(\partial^{\alpha}\phi - \partial^{\alpha}t)]$$

The regime of validity for the EFT is going to be between $\tilde{\mathcal{K}} << \tilde{\lambda} << 1$. We will examine this in more details in the next section.

The term corresponding to the tension of the operator as measured by gravity will be $\frac{\alpha^2\mathcal{K}}{12\lambda}(h_{tt})$. This is called the Tadpole term. The tadpole term will measure the emission of a graviton. We need the graviton in the bulk to actually measure the tension for this defect. If we take $\lambda \to 0$, this term would become infinitely heavy. If we try to decouple [For $\lambda \to 0$, $\mathcal{K}$ should also need to be zero, hence decoupling] the collective coordinate , it makes the object strongly coupled. From the bulk, this would be signaled by a black-hole formation. In other words, going to zero would lead to infinte heaviness and thus a black hole would be formed and we would be moving beyond the suggested validity of the EFT regime. We need to measure the graviton so that we don't push it too close to zero or remove it.

Choosing the field profile , close to zero but not non-zero , the total effective action can be written as:- $S = S_{bulk} + S_d + S_G + S_{graviton}(t, \mathcal{K})$. S_{bulk} is the original bulk action. [14]. The second and third factors recover the defect action as derived in the previous sections.

The final factor $S_{graviton}(t, \mathcal{K})$, consists of all additional terms containing the graviton field that act as corrections to the Goldstone action. The corrections from $S_{graviton}(t, \mathcal{K})$ will modify this coupling and allow us to define an effective tension in the worldvolume theory, similar to the previous sections.

As the width λ decreases , the energy density increases and perturbation theory breaks down as the defect experiences stronger gravitational interactions. There are no topological sectors at high energies.

So , in conclusion, the idea which we are trying to showcase here is the recovery of the Topological Operator. One of the ways to do it is to take λ to zero to be consistent and the second way is to show that the Goldstone modes decouple or freeze. And then, a leftover piece can be removed by a counterterm. The presence of gravity obstructs $\lambda \to 0$, [11] so in Field Theory if we wish to study this operator, we need to be able to regulate it. The choice of the regulator also has to be important, because it will eventually take us to the topological limit.

6 Induced EFT , Species Scale and Swampland

In the previous section, we saw that the presence of gravity obstructs $\lambda \to 0$, and hence there is a need to be able to regulate it.

In the context of Effective Field Theories (EFTs), the situation becomes more delicate when a large number of charged species (a detailed description of this will be provided in this section)are introduced: each new species contributes degrees of freedom, and beyond a certain point the EFT risks becoming inconsistent with gravitational constraints.

An induced EFT refers to an effective description obtained after integrating out high-energy degrees of freedom. Such an EFT is valid only below a certain cutoff, where heavy states have been removed but their residual effects persist through higher-dimensional operators. Since we are working near the Planck regime, the reduced Planck mass defined as $M_{pl}^2 = \frac{1}{8\pi G_N}$

[11]The presence of gravity obstructs the existence of an ideal Topological Operator or topological phases. In a system without gravity, a zero-width limit $\lambda \to 0$ can be taken to recover topological properties, but gravity makes the limit ill-defined, which prevents the topological operators from existing in their pure form

We must recognize an essential feature of quantum gravity: exact global symmetries are forbidden — they must either be broken or gauged. This forces us to frame EFTs within a cutoff structure, where quantum gravity effects become significant well below the naive Planck scale.

The natural regulator for this cutoff is the species scale. A scale where EFTs can be induced from and from where Topological sectors(defects, anomalies etc) can also be recovered. The species scale is the energy threshold where the multiplication of these species, given by N lowers the effective scale of quantum gravity.

$$\Lambda_{QG} \lesssim \frac{M_{pl}, d}{N^{\frac{1}{(d-2)}}}$$

In a d- dimensional EFT, weakly coupled to Einstein gravity, the species scale is given by Λ_{QG} . [16] N is the number of species with masses below such UV-cutoff and $M_{pl,d}$ denotes the d-dimensional Planck mass.
This is the point where an EFT can be said to be induced: the low-energy theory emerges after integrating out heavy states, but it also retains traces of UV physics in the form of higher-dimensional operators and topological sectors (such as defects, anomalies, or nonlocal operators).

Thus, the species scale sets the boundary where an induced EFT remains consistent with gravity while still encoding information about the topological and anomaly structures inherited from the UV completion.. The derivation of the species scale is shown below.

6.1 Derivation of the Species Scale

In the framework of EFTs when we are trying to compute any observables which could be measured in experiments or in order to explain any physical phenomenon , it is necessary to first explain the spectrum of the theory, namely the number of species, their quantum numbers, their interactions etc. Moreover, the regime of validity as mentioned in the previous section also needs to be specified. This is important because we have to understand the precise description where breaks down and also as a possibility, another one takes over. The latter part of the information is usually taken over by two seemingly unrelated quantities, Λ_{UV} and Λ_{IR}. [16] Λ_{UV} sets the maximum energy density of any field configuration described by the EFT, while the latter imposes the maximum wavelength in the theory and hence it has to do with the size of the region of spacetime within which the theory is restricted to.

6.1.1 Species Scale Formalism

The goal is to formalize a EFT-gravity lagrangian. And from this formalism derive the species scale. An Effective Field Theory(EFT) Lagrangian is essential to describe the dynamics below the UV-cutoff. If we were to consider Gravity as a dynamical term, then we need to couple the EFT consistently with General Relativity. In induced EFTs as mentioned above, the residual effects persist through higher dimensional operators. These higher dimensional operators essentially carry the "memory" of the Planck scale as interactions. These do not matter at everyday energies, but become significantly useful as one approaches the cut-off where EFTs break down. Let's divide this into a few parts.

* Basis for the Lagrangian
* Higher Dimensional Operators.
* Components of the Lagrangian.
* An EFT-gravity Langrangian ($\mathcal{L}_{EFT+Grav}$).

1. Basis for the Lagrangian

The fundamental basis for our lagrangian would be a total action S of a point -particle which is the combination for both the Einstein-Hilbert Action(for low-energy interaction) as well as the dynamics of the graviton. It should be given by:- $S = S_{EH} + S_{pp}$, where S_{pp} represents the action of the point-particle (dynamics of the graviton). [17]

It is possible to calculate this equation by expanding the metric around the flat space as mentioned in the section(5.4) by $g_{\mu\nu} = \delta_{\mu\nu} + \mathcal{K}h_{\mu\nu}$ Once we integrate out the graviton field to obtain an effective action for the co-ordinate alone would be:-

$$exp[iS_{eff}[x^0]] = \int \mathcal{D}h_{\mu\nu} exp[iS_{eff} + iS_{pp}]$$

The effective action $S_{eff}[x^0]$ has a real part that generates the coupled equations of motion for the point-particle . It also has an imaginary part which measures the total number of gravitons emitted by a fixed particle configuration - collective coordinates x^a over a large arbitrary time $T \to \infty$. Hence the overall picture should resemble something like :- $S_{eff} = \int d^4x\sqrt{-g}\mathcal{L}_{eff}$. This effective actions consists of both matter and gravitational terms.

2. Higher-Dimensional Operators

The residual Planck effects persists through these operators (non-renormizable group). These encode the information from the heavy physics that has been integrated out. They can also play a role in anomaly cancellation as well [12].

- Purely gravitational Operators

$$\mathcal{O}_1 = R^2, \quad \mathcal{O}_2 = R_{\mu\nu}R^{\mu\nu}, \quad \mathcal{O}_3 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \quad \mathcal{O}_4 = R^3$$

- Matter Gravity Couplings

$$\mathcal{O}_5 = R\phi^2, \quad \mathcal{O}_6 = R_{\mu\nu}\bar{\psi}\gamma^\mu D^\mu\psi, \quad \mathcal{O}_7 = RF^{\mu\nu}F_{\mu\nu},$$

- Higher Derivative matter operators

$$\mathcal{O}_8 = \frac{1}{M_{pl}^2}(\partial_\mu\phi\partial^\mu)^2, \quad \mathcal{O}_9 = \frac{1}{M_{pl}^2}(\bar{\psi}\gamma^\mu D_\mu\psi)^2$$

Each $\mathcal{O}_i$ respects diffeomorphism invariance(general covariance) otherwise it would be inconsistent with quantum gravity.

3. Components of the Lagrangian

The goal here is to treat General Relativity as a low-energy EFT with higher-derivative corrections suppressed by M_{pl} If we consider General Relativity and its quantization, the Einstein action can be given by:-

$$S_{grav} = \int d^4\sqrt{-g}[\frac{2}{\mathcal{K}^2}R]$$

where $\mathcal{K}$ as mentioned before is $32\pi G_N$. the induced metric g is the determinant of the gravitational metric $g_{\mu\nu}$, $g = detg_{\mu\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$

Heavy spinless matter fields interact with the gravitational fields as described by the action. [18]

$$S_{matter} = \int d^4\sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2]$$

The guiding principle underlying general relativity is that of gauge symmetry, i.e., the local invariance under coordinate transformations. This forces the introduction of geometry, and requires us to define the action of the theory using quantities invariant under the general

[12] Higher Dimensional operators aid in anomaly cancellation by introducing Wess-Zumino terms- cancellation of fermionic anomaly, chern-simons terms in odd dimensions $11D$ or $5D$ Supergravity

coordinate transformations. However, this isn't sufficient to completely define the theory as many quantities are invariant.

$$S_{grav} = \int d^4x\sqrt{-g}[\Lambda + \frac{2}{\mathcal{K}^2} + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + ...]$$

where the constants denote the higher powers of R, $R_{\mu\nu}$ etc, upon expansion. Each term in the action are separately invariant under general coordinate transformations. This equation represents the classical gravitation action where the quantities are invariant. [19]

4. The EFT - gravity Langrangian ($\mathcal{L}_{EFT+Grav}$)

In the Einstein Hilbert action, we have to encode the spacetime geometry represented by the Ricci Scalar R. [20]. The Einstein-Hilbert Action which is given by:-

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4\sqrt{-g}R$$

Now , considering the Planck mass $M_{pl}^2 = \frac{1}{8\pi G_N}$, we can rewrite the Einstein Hilbert action as:-

$$S_{EH} = \int d^4\sqrt{-g}\frac{M_{pl,d}^2}{2}R$$

The lagrangian from S_{grav} can be written as:-

$$\mathcal{L}_{grav} = \frac{M_{pl,d}^2}{2}R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + ...$$

The matter-EFT Lagrangian(scalars, fermions, gauge fields) will be:-

$$\mathcal{L}_{EFT}^{matter}(\phi, A_\mu, \psi) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial\phi)^2 - V(\phi) + \bar{\psi}i\not{D}\partial\psi + ...$$

Hence, the total EFT-gravity Lagrangian can be described as:- $\mathcal{L}_{grav} + \mathcal{L}_{EFT}^{matter}$

$$\mathcal{L}_{EFT+grav} = \sqrt{-g}[\frac{M_{pl,d}^2}{2} + \mathcal{L}_{EFT}^{matter}(\phi, A_\mu, \psi; g_{\mu\nu)} + \sum_i \frac{c_i}{M_{pl,d_i}^{d_i-4}}\mathcal{O}_i]$$

This is quite similar to the effective action S_{eff} described before, after you break down all of the components of the Lagrangian indiviually. c_i are dimensionless coeffients , multiplying higher dimensional operators. It is important to note here that we are replacing the flat Minkowski metric with the gravitational metric $g_{\mu\nu}$ in the matter Lagrangian. The mathematical implication of a $4D$ EFT is that there will be no suppression(i.e, it belongs to the matter sector of the EFT), when $d = 4$, $\lambda = 1$. If the value of d is less than 4 , then we will get relevant operators which grow in the IR range. If $d > 4$, in that case we will have the higher dimensional operators which encodes information related to heavy physics.

Species scale from Higher curvature corrections

Consider a general theory of Einstein gravity in d-dimensions coupled to a massless scalar field with two-derivative action.

$$S_{eff} = \int d^dx\sqrt{-g}(R - \frac{1}{2}(\partial\phi)^2)\frac{M_{pl,d}^{d-2}}{2}$$

Generally speaking, this effective action comprises of corrections by higher -derivative terms that encode the effects of heavy physics(the effects of Quantum Gravity at the EFT level). These corrections can be parametrerized as:-

$$S_{correc} = \frac{M_{pl,d}^2}{2}\int d^dx\sqrt{-g}(\sum_{n=2}^{\infty} a_n(\phi)\frac{\mathcal{O}_{2n}(R)}{M_{pl,d}^{2n-2}})$$

where $\mathcal{O}_{2n}$ are dimension $2n$ operators formed from contractions of the Riemann tensor R. In this case, the coefficients $a_n(\phi)$ encode the information about the UV completion of the effective theory of gravity [21]. If all these coefficents were independent of the scalar field ϕ and were of order $\mathcal{O}_1$, the expansion of the effective gravity would break down at curvatures of the order of Planck. Hence, there is a necessity to formulate a scale below the naive planck scale . This is called the species scale and is given by Λ_{QG}

$$a_n(\phi) \leq (\frac{M_{pl,d}}{\Lambda_{QG}(\phi)})^{2n-2}\hat{a_n}$$

where $\hat{a_n}$ is a theory dependent Moduli -independent constant.

6.2 Swampland

So far we have seen that Λ_{QG} is the natural cut-off for a theory of species that is weakly coupled to Einstein's gravity. However in consistent theories of gravity , we typically expect that there are additional light-states beyond the massless level. We know that EFTs always break down at a particular cut-off scale and in order to go beyond this scale, one should include new degrees of freedom. The new degrees of freedom should be such that they do not lead to inconsistent theories of Quantum gravity. It is this exact conjecture that leads us into swampland. This is one of many conjectures in swampland.

Recent years have seen the emergence of a new picture, a new paradigm for quantum gravity. It has become clear that certain low energy theories are that seem consistent from several differernt viewpoints (anomaly cancellation can be used as an example) , cannot be coupled to quantum gravity in a consistent way. [22] The low energy theories (plausible looking EFTs) that cannot be consistently coupled to gravity are said to belong to the swampland.

In string theory, string compactifications are used to compactify from 10 dimensions to 4 dimensions. Such compactifications may give rise to an enormous number of ground state solutions, commonly referred to as the landscape of string theory. In the space of consistent low energy EFTs, the border separating the landscape from the swampland is delineated by a set of conjectures on the properties that those theories should have / avoid in order to allow a consistent completion into Quantum Gravity. One of the most fundamental of these conjectures is the statement that Quantum Gravity does not permit Global Symmetries. They must either be gauged or broken at high energies. [22] There are others as well like WGC(Weak Gravity Conjecture) and De-Sitter Conjecture, but those are outside the scope of this paper.

One of the implications of a EFT breaking down is the emergence of infinite towers of light particles. If N increases, then the species scale cut-off would decrease. But since there is an infinite tower of light particles, the question is - are there light particles that have masses below the species scale Λ_{QG} or maybe even much smaller. The answer is **YES**. The point of the species scale isn't that such particles are forbidden , but we cannot trust the EFT description of gravity with all of them simultaneously above the cut-off. In a consistent theory of Quantum Gravity(eg:- string theory) , the full theory is UV complete such that there is no paradox in having such large set of particles.

What the species scale tells us is that the low-energy EFT of gravity plus these species cannot be extrapolated to high energies. Instead, new physics appears already at Λ_{QG}. Let's understand this with an example.

Suppose, we have an EFT of gravity with N light species (say scalars, fermions, each with mass $<< M_{pl}$. Naively, one may think that that this EFT should remain valid up to the Planck scale $\Lambda_{naive} \sim M_{pl}$. But this assumption is inconsistent once we include the effects of Quantum Gravity. A Black-hole of radius R has an entropy(Bekenstein-Hawking) of $S_{BH} = \frac{A}{4G_N} \sim (R.M_{pl})^2$. An EFT with N light fields contributes N species of Hawking radiation modes. Let Λ_{BH} be the cut-off of the EFT. Consistency requires $S_{EFT} \lesssim S_{BH}$. The corresponding species bound $\Lambda_{BH} \lesssim \frac{M_{pl}}{\sqrt{N}}$ suggests that the gravitational EFT cannot be trusted beyond Λ_{BH} species. The EFT predicts too much entropy compared to the maximum amount allowed by the Black Holes.

This is exactly the kind of inconsistency that Swampland seeks to exclude. So to sum it all up:-

- Landscape (consistent):- Gravity + many species , but EFT cut-off lowered to Λ_{QG}.

- Swampland(inconsistent):- Gravity + many species , but it is still assuming the cut-off at M_{pl}.

6.3 Swampland Distance Conjecture

The breakdown of EFTs leads to the emergence of infinite towers of light particles. Now, thinking about the problem in terms of distance, is it possible to recover topological operators or Goldstone modes/sector when we move far away from the source field and make use of higher dimensional operators ?. This is where the Swampland Distance Conjecture comes in. [23] Suppose, we have a moduli space(field space) of scalar fields. When we move very far into the moduli space, an infinite tower of states becomes exponentially light, causing the EFT to break down.

Now, consider a gravitational effective theory defined within a moduli space. We are considering a space which is parameterized by the massless scalar fields in the theory. The metric of the theory is given by the Kinetic form of the scalar fields. Starting from a point P in the moduli of the scalar field and moving towards a point Q , an infinite geodesic away (i.e, $d(P,Q) \to \infty$) , one encounters an infinite tower of states which becomes exponentially massless with the geodesic distance.

$$\frac{M(Q)}{M_{pl}} \sim \frac{M(P)}{M_{pl}} e^{-\alpha d(P,Q)}$$

α is the order one constant in Planck Units.

To understand the Swampland Distance Conjecture(SDC) [23] and develop some intuition on it , let's consider a canonical example. [22] Let's consider a theory which is compactified on a circle of size R. It is well known that in the Kaluza Klein (KK) modes in such a circle compactification has a mass that scales with the internal radius as:-

$$m_n^2 = \frac{n^2}{R^2}, n \in \mathbb{Z}$$

where m_n^2 is the KK tower mass. We can consider the distance between two points in the field namely, R_i and R_f is measured by the field space metric given by $1/R^2$ and it yields

$$d(R_i, R_f) = \int_{R_i}^{R_f} \sqrt{\frac{1}{R^2}} dR = [log(R_f/R_i)]$$

Therefore, starting at any finite radius, there are two points that live at infinite proper distance namely $R \to \infty$ and $R \to 0$. Approaching at $R \to \infty$, it is quite evident that the KK tower mass equation predicts the behviour described by SDC. Namely,

$$m_n \sim e^{-d(R_i, R_f \to \infty)}$$

On the other hand, approaching the infinite distance point $R \to 0$, one could be tempted to say that SDC is violated. However, this isn't the case if we were to consider string theory, which includes an infinite tower of winding states[13] with masses given by:-

$$m_\omega^2 = \frac{\omega^2 R^2}{(\alpha')^2}$$

These become exponentially light in terms of the field space distance d as R approaches the zero-limit. The SDC can be understood as a restriction on the range of validity when EFT is coupled to gravity, in the sense that an EFT defined at a point in moduli space cannot be extended to a point which is at an arbitrarily large distance from the initial one . If one tried to do so, an infinite number of light degrees of freedom would become light and break the aforementioned EFT description.

So, in terms of the Goldstone modes which is the main focus of this discussion, as we try to make the Goldstone sector extremely light or its range large , we tend to pull down the species scale Λ_{QG}, so that the higher dimensional operators that we neglected become important and the EFT ceases to be predictive before we reach the Planck scale.

6.4 Goldstone modes and Large Field Spaces

A goldstone boson as we know arises when a continuous global symmetry G is spontaneously broken into a subgroup H. The mathematical proof of which is described in this paper [6]. This leaves the space G/H as the target space for the Goldstone fields.

[13] As we move towards infinite distance, in a theory's fundamental space, a series of states , each "winds" down around some dimension, becoming infinitely light

Now, moving far into the Goldstone field space, corresponds to moving far in moduli space of the underlying quantum gravity theory. For non-compact target spaces - Large excursions are exactly the infinite-distance limits where the SDC predicts a tower of states becoming exponentially light. For compact spaces,($n \in \mathbb{Z}$) (S^1) however, there is an infinite distance limit where KK/winding states descends. (their mass scale drops below the cutoff, and they must be included in the low-energy effective theory.) For a scalar field ϕ , if we move a geodesic distance $\Delta\phi$, then the tower would be:-

$$m_n \sim m_p e^{-\alpha \frac{\Delta\phi}{M_{pl}}}$$

where m_p is the mass at the starting point. This assumption flips however when we consider the behaviour at the Planck scale.

The Goldstone manifold's curvature matters: If its radius is approximately close to M_{pl} or smaller, quantum gravity corrections are huge and we cannot treat the Goldstone mode as a weakly coupled degree of freedom.

Towers of light states can mix with the Goldstone, potentially turning it into part of a higher-dimensional gauge field or a string winding mode.

So in conclusion SDC can be used to generalize the appearence of emergent light particles and in turn we get to analyse the nature of the Goldstone sectors in a far moduli space . For a theory with N species that is weakly coupled to Einstein gravity, the infinte tower of states becoming light asymptotically is comprised by KK modes or by the massive excitations of the critical string states, respectively. This allows us to be able to compute the species scale in the presence of such towers.

7 Recovery Of Topological Operators

When an EFT breaks down , the missing degrees of freedom often reorganizes into non-local topological structures. If we integrate out massive charged fields , we often generate topological operators.(Wilson loops, 't Hooft lines) in the IR EFT. They basically encode the memory of heavy charges even if the charges are no longer dynamical.

The logic on which we are going to be working upon is -

If there are no dynamical charges of charge q in the theory, a Wilson line cannot terminate — it must be a closed loop (or extend to infinity). That makes it topological. (The correlator must remain unchanged). Also,

If there are dynamical charged particles of charge q, the worldline of such a particle can act as the endpoint of W_q. In the path integral configurations where a particle worldline starts (or ends) on the Wilson contour are allowed. So, the Wilson operator is not protected from being cut: it can be terminated by dynamical charge creation. The deformation test is done by the ratio of insertions.

Let's consider the example of symmetry operators for an electric 1-form. We are using $4D$ Maxwell's theory, the action of which is given by:-

$$S = \int -\frac{1}{2e} f \wedge \star f = \int d^D x - \frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} = \frac{1}{4e^2} \int F \wedge \star F$$

For a symmetry operator given by:- $U_\alpha(\Sigma)$, if deformed by $\Sigma^{'}$, the ratio of the insertions would look like $U_\alpha(\Sigma)/U_\alpha(\Sigma^{'})$

We can prove that the correlators are independent of smooth deformations i.e, they remain topologically invariant from ward identitis. The ratio of deformations over a volume V would be:-

$$\frac{U_\alpha(\Sigma^{'})}{U_\alpha(\Sigma)} = exp(i\alpha \int_V d(\star F)) = exp(i\alpha \int_V \star j)$$

using Maxwell's equations $d \star F = \star j$. If no dynamical charges are present in the EFT, the correlators are independent of smooth deformations. Since the missing degrees of freedom reorganizes as non-local topological structures, we have to ask whether the symmetry form can end on dynamical charges. Hence, we can use Wilson correlators and the resultant linking phase will prove whether or not the charge is measured.

From a Wilson line $W_q(C)$, the joint insertion of the line and the symmetry operator that represents a holonomy A over a closed manifold C can be written as:-

$$\langle U_\alpha(\Sigma) W_q(C) \rangle = \int \mathcal{D}A e^{iS[A]} exp(i\alpha \int_\Sigma \star F) exp(iq \oint_C A)$$

Unlike what we had shown in the section of Higher Form Symmetry in 4D Maxwell's theory, where the expectation value represented the worldine of a charged particle which had no dynamics, here, the charge has to be measured, since we are trying to determine the dynamical/non-dynamical nature of the charge. The phase that arises because of the linking number which describes the joint insertions, measures the charge q.

$$\langle U_\alpha(\Sigma)W_q(C)\rangle = e^{iq\alpha L_k(\Sigma,C)}\langle W_q(C)\rangle$$

This shows that U_α acts as the generator of the electric 1- form symmetry where the charge q is a dynamical quantity which is measured by the Linking number. This paper [24] provides a comprehensive explanation on 1- form symmetry.

We already know that the higher derivatives in the effective action of the EFT encode the heavy physics. Its prominent when $d > 4$. We are interested in what "emerges" after we integrate out these heavy charges. When we say "Emergent" , it means "after integrating out heavy charges."

If we start from the full path integral with massive matter $\mathcal{X}$, we have a field $\mathcal{X}$ with mass M charged under the gauged field A_μ, a question that arises is - does the Wilson line end freely at some point ?.

7.1 Introducing a Limit

At energies $E << M$, (E is the effective field theory energy) processes that would create a pair $\mathcal{X} - \mathcal{X}'$ pair, are exponentially suppressed. (probability (or amplitude) for certain processes — like pair-creating a heavy charged particle to cut a Wilson line). Thus, the Wilson line cannot be fully terminated by the dynamical $\mathcal{X}$ worldlines. [[25], [26]] In a full path integral representation , the correlators involving Wilson lines , include a sum over all possible worldine configurations of $\mathcal{X}$. [[27], [28]]

$$\mathcal{Z} = \int \mathcal{D}A\mathcal{X}e^{is[A]+iS_{matter}[A,\mathcal{X}]}$$

As you can see, this is quite familiar to the expectation value we had derived after combining equation(5) and equation(6). In that section we had coupled the gauged term with a non-dynamical charged term . Since the objective here is to find a limit to the dynamical worldine and also measure the dynamical charged term , we are allowed to place a limit to control the deviation.

If $\mathcal{X}$ is very heavy, [[29] , [24]] contributions where a $\mathcal{X}$ worldine ends on ∞ are suppressed by $exp(-ML)$, with L being the worldline length. At $M \to \infty$ limit, such contributions vanish. At long distances, the 1- form symmetry emerges as an IR symmetry. When $\mathcal{X} - \mathcal{X}'$ pair production is allowed, the symmetry operator U_α is no longer topological. The estimated amplitudes here will spoil the topological invariance. $\langle W_\alpha \rangle \sim e^{-ML}$ controls such deviations.

In other words , the dynamical charges spoil the exactness of the symmetry. As long as there are no dynamical charges, U_α will remain topological. Introduction of gravity though, will completely spoil the story.

So, what can we conclude for here ?. Let's bring it back.

It is already known that the physical interpretation of a Wilson line operator represents the worldline of a probed charged particle, this means a particle which has no dynamics. The higher derivatives in the effective action of the EFT encode the heavy physics, so missing degrees of freedom can be re-used. When massive , charged fields are integrated out , topological operators emerge. But there's a catch to this type of emergence. In the absence of dynamical charges, a Wilson line cannot terminate. On the contrary, if there are dynamical charges , the worldine of such a particle can act as the endpoint of the Wilson line. We can measure the nature of such a charge by introducing a deformation on Σ. The Linking number determines this.

Now, for a heavy field $\mathcal{X}$ with mass M , the question is- in the presence of dynamical charges , can the Wilson line end at some point ? . The answer is **YES**. We have to take into consideration , all possible worldines that are available for the field within a certain energy limit i.e, $E << M$.

If the field is very heavy, all contributions by the field can be suppressed by introducing a $M \to \infty$ limit. At long distances, IR symmetry emerges. But, at the same time, we need something to control the deviation , otherwise a field pair production will occur , which is going to cut the Wilson line and the topological identity of U_α will disappear. This can be done by introducing $exp(-ML)$. [[30], [29]] . This term will suppress the contributions close to ∞ where the worldine is supposed to end. The exponential term given by:- $\langle W_\alpha \rangle \sim e^{-ML}$, whose amplitudes are able to preserve topological invariance (topological identity- suppressing the breaking of topological invariance caused by heavy charges.) , by controlling the deviations.

In the full picture, topological symmetry emerges in the IR because violations are exponentially suppressed. $\langle W_\alpha \rangle \sim e^{-ML}$ is a Boltzmann like suppression.

So, we can conclude that - Thus, for energy scales $E << M$, the dynamical effects of $\mathcal{X}$ are negligible, and the Wilson line behaves as if it were topological. In this way, topological invariance emerges in the IR as an approximate symmetry, with violations exponentially suppressed by the heavy mass. At infinite mass, the topological identity is exact. At finite but large mass, deviations exist but are tiny.

8 Summary

In this paper, we started off by explaining the basics of generalized global symmetries. The movement of a local operator O in spacetime (x, t) , justifies the action on the Unitary Operator $U(t)$. So, when a system has properties which do not deform under any change, we call these properties as topological operators. We have also understood that a Unitary Operator can be generalized to a co-dimension 1 surface. The symmetries described are the ordinary global symmetries whose topological operators, are associated with a codimension 1 surface. For higher-form symmetries these topological operators are associated with a co-dimension of $(q+1)$ surfaces. We move on towards topological operators in QFT, where we demonstrate that for a topological operator, defined by a charge operator - the correlation functions remain invariant under small deformations. Charges are very rarely shown as operators, but this concept is crucial in the description of generalized global symmetries where charges are no longer dynamical/non-dynamical , but extended objects/operators (lines, surfaces etc). We are interested in symmetry operators that remain independent of the Lagrangian and the charges and charged objects are categorized as operators.

The symmetry operator's dependence on a manifold is topological. Also, the symmetry transformations will have to satisfy the group law. The operator remains unchanged even when the manifold is deformed slightly. In other words, the symmetry operators are a special kind of topological operators which preserve topological invariance.

When another operator is charged under symmetry, only then do we see small deformations on the operator. The charged object of a q- form global symmetry can be represented by an operator $V(\mathcal{C}^{(q)})$. A concrete example which can be used is that of $U(1)$ symmetry. When $G = U(1)$, there will be a conserved $(q+1)$ -form Noether charge j. For a continuous symmetry $G^{(0)}$, we can define a topological operator that defines the symmetry action on the operators. Here, the charge remains invariant under the deformations of the manifold which is consistent with our original conclusion with respect to topological invariance. Higher form symmetries can similarly be proven using $4D$ Maxwell's theory. But there's a caveat.

When it comes to extended operators, on a closed surface, we can construct local operators called Wilson lines that are gauge invariant. The physical significance of such operators become prevalent when we are considering the dynamics of the charge. Considering a conserved current that is moving along a curve, the Wilson line can be written as $W_{q_e}(C)$, where q_e is the charge of the Wilson line. Basically, in short, for 0- form symmetries, Wilson loops gives us the physical interpretation of point-like observables on a closed loop (contour).

This motivates the question - what happens in higher form symmetries ?. Whenever we try to couple the global symmetries to a background gauge field(insertion of the topological defect), it becomes impossible to preserve the invariance of the action under gauge transformation. As soon as a background gauge field is introduced, the symmetry breaks. A theory is said to have 't Hooft anomalies when this happens. Now, anomalies can be cancelled by introducing anomaly inflows from a classical field theory in one higher dimension. If that is the case, an anomaly can be characterized as an action of the theory. As we have written in section (4.2), if we have a d- dimensional theory, that has a $U(1)$ global q - form symmetry , then the probable 't Hooft anomalies that correspond to topological action in $(d+1)$ dimensions , built of out $(q+1)$ - form on a closed manifold C, can lead to classical theories, even though they might not be exact theories of quantum gravity. This automatically leads us to questioning what happens to topological operators when gravity is introduced, which can be investigated for both ordinary symmetries and higher-form symmetries. Can we construct an effective action that includes gravity when topological defects are introduced ?.

From a broader perspective, we are interested in understanding how topological excitations can combine under a topological order that can represent an algebraic structure. We need to provide a regulated version of the operator which can aid our recovery of the operator later on. So, what exactly happens when we insert an operator when we have fields in our system ?. We know that QFT encounters issues relating to singular insertions, hence there is a need for regulation. by giving the field a width of let's say λ , we are regulating it in some way. We are treating the topological defect as a solitonic object. Upon the insertion of a defect, wiggles are going to emerge as a consequence. The wiggles are the Goldstone modes. This phenomenon can be studied by introducing a shift in the scalar field by α, $\phi \to \phi + \alpha$.

In this situation, an important element that needs to be taken into consideration is the limit of the width λ. We know that topological operators expressed in terms of currents need regularization as expressed in section(5.2), but how do we freeze the limit so as to recover the topological operator. Basically, we are looking for the effective action of the goldstone mode S_G upon introducing a topological defect.This can be accomplished by introducing a zero-width limit, $\lambda \to 0$, as a consequence of which the collective co-ordinates will freeze and provide a regulated version of the topological operator $U(\Sigma)$. By treating the topological defect as a solitonic object that is regulated under a field profile configuration, we have derived an effective action S, which is a combination of the bulk action, the regulated defect as well as the Goldstone modes.

An immediate question is - can we do the same in the in the presence of gravity ?. We would have to take into account the bulk theory coupled to gravity. The gravitational metric would take the form $g_{\mu\nu} = \delta_{\mu\nu} + \mathcal{K}h_{\mu\nu}$ where $\mathcal{K} = 32\pi^2 G_N$ is the perturbative parameter. Now, expanding upon smaller fluctuations, we can receive an effective action. If we decide to shift the scale , once again by α, we are going to receive some total effective action of a non-trivial EFT, which includes the bulk term coupled with the effective action of the graviton $S^{\lambda}_{graviton}$. The tadpole term will measure the emisson for the graviton. If we wish to decouple , i.e, $\lambda \to 0$, the perturbative parameter , will also be zero and hence the object would become infintely heavy under strong coupling and thus signal the formation of a black hole. Hence, we can arrive at the conclusion that a field profile needs to be chosen in a way, such that it is close to zero and not non-zero. A total effective action can be derived which is the combination of the action of the graviton, the total bulk action, the action of the defect and the Goldstone action. As the width λ decreases, the energy density increases and perturbation theory breaks down as the defect experiences stronger gravitational interactions.

Since, the presence of gravity obstructs $\lambda \to 0$ we can thus conclude that a topological limit can be established if we introduce a regulation. When the width of a regularized operator approaches zero, the fluctuations freeze, leading to the properties of a true topological operator.

After deriving the effective action of a non-trivial EFT that couple the graviton with the Goldstone mode, we realize that the regime of validity of the said EFT , needs to be within a perturbative limit. Then, a fundamental question that arises regarding the limit to which an EFT will remain consistent, before becoming inconsistent when gravitational constraints are applied. We discuss the species scale, which acts as a natural cut-off for EFTs where Quantum Gravity effects become significant. This scale is well below the naive Planck scale. This can be described as the cut-off structure within which EFTs need to be framed. For a d-dimensional EFT , weakly coupled to Einstein gravity , the species scale is given by Λ_{QG}. The species scale sets the boundary, where an induced EFT will remain consistent with gravity , while retaining information about topological structures.

The species scale formalism is described by an EFT Lagrangian, which is essential to describe the dynamics below the UV-cutoff since we are interested in using residual degrees of freedom. The perturbative regime of validity is also essential since it gives us the range precisely in which one description of the theory breaks down and where a new one takes over. It's given by Λ_{UV} and Λ_{IR}. One of the implications of EFTs breaking down is the emergence of infinite towers of light particle states, the question that arises here is that - are there light particles whose masses exist below the species scale or even smaller ? and more importantly - are they consistent ?. We already know that Global symmetries are inconsistent with Quantum Gravity theories, so these are exactly the kind of inconsistencies that Swampland Conjecture seeks to exclude. The Black-Hole entropy example provided in section(6.2) demonstrates this.

If we think about in terms of distance - If we extend the moduli space, the infinite tower of states becomes exponentially light , causing the EFT to break down. We need to put a limit on the moduli space within which the EFT coupled to gravity can be restricted within the regime. The Swampland Distance Conjecture puts that restriction. This enables us to analyse the nature of the Goldstone sector given large moduli spaces. Moving on , we need to figure out a way to recover topological operators and preserve topological invariance, so as to gain new insights into symmetry-violation in gravitational theories.

As mentioned in [14] , a potential implication is that the would-be symmetries in QFTs that are coupled to gravity, without gauging, imply the existence of dynamical defects whose physics is in principle accessible within perturbation theory. This may lead to new insights into symmetry-violation in gravitational theories. Basically in short , suggesting that would-be symmetries are not exact once coupled to gravity, because defects become dynamical i.e, the pair production of such QFTs would get cut by the Wilson line.

We have shown in this paper that for heavy fields $\mathcal{X}$ with mass M, contribution from the field that is close to infinity can be suppressed. Within energy scales , $E << M$, we can introduce a limit, $exp(-ML)$, where L is the length of the worldline , that can control the deviations and are able to preserve topological invariance. As mentioned in section(7.1), the emergence of such topological invariance happens in the IR range as an approximate symmetry , with violations of the symmetry being exponentially suppressed by heavy mass. At finite but large mass, the deviations that exist are tiny . Further study into the measurement of dynamical charges can give insights into the physics of new and emergent symmetries that can preserve topological invariance at large but finite mass.

The Boltzmann like suppression $\langle W_\alpha \rangle \sim e^{-ML}$ controls the deviations and provides a natural mechanism for the emergence of topological invariance in effective theories, the heavy dynamical charges regulate the breaking of symmetry, but leave behind an IR remnant that looks like a topological operator.

Further exploration of the dynamics of such heavy defects could shed light on how would-be global symmetries in QFT reappear as emergent, approximate symmetries when coupled to gravity.

9 Future Work and Conclusion

So, starting from generalized global symmetries, symmetry operators , we have described the fate of topological operators when they are embedded with gravity. Using missing degrees of freedom(with gravitational constraints) in EFTs we make our way back towards the recovery of topological operators and investigate how to preserve topological invariance given different mass limits.

For future work one might extend the idea of EFTs coupled with gravity to background independence. Roughly speaking, the term background independence refers to the realization that spacetime quantities are dynamical by nature and are not put in by hand. There exists no absolute structures or fixed fields, any metric structure should be dynamical in nature and not externally put in by hand. The physical quantities are diffeomorphism invariant. Background metrics can be re-interpretated in EFTs. It is Diffeomorphism invariant, so no explicit background structures in the EFT is allowed beyond what is dynamical. A regulator can also be chosen , so that background independence cannot be broken too severely. Another future prospect that will be explored in subsequent papers is whether Supergravity can help smooth out higher form global symmetry divergences. We know that Higher-form symmetries can lead to divergences in charges: e.g. infinite Noether charge falling into a black hole. In the path integral formalism, these show up as divergences associated with summing over sectors(large Wilson surfaces). Supergravity has several built-in mechanisms that remove or regulate divergences from higher-form global symmetries: As discussed in section(4.2), anomalies and inflow being one of those mechanisms, Brane absorption is also one of them . Supergravity multiplets can also emerge from induced EFTs. There are a few ways to do this. From string theory, we can integrate out the heavy modes $\rightarrow$ supergravity multiplets in 10D/11D, very similar to what we have discussed about integrating out heavy fields in section(6.1). There are other methods as well.

The topics discussed in this paper presents a comprehensive overview on generalized global symmetries, symmetry operators as well as higher form symmetries. The fate of topological operators in the presence of gravity is an integral part of our discussion . In the later half, we investigate deep into the species scale, it's derivation, induced EFTs, the Lagrangian formalism of Gravity coupled with EFTs. As a consequence of our exploration into EFTs, consistent and inconsistent theories of gravity, we briefly touch upon Swampland and Swampland Distance Conjecture . From missing degrees of freedom in EFTs, we figure out a way to not just recover topological operators, but also to place a limit within which consistent theories can emerge that preserves topological invariance within a perturbative range and also help in analysing symmetry-violations and the reappearence of global symmetries as emergent , approximate symmetries when coupled to gravity.

10 Appendix

10.1 I. Higher representation

For $q = 0$, we say that 0 charges of a 0 - form symmetry group are the representations of $G^{(0)}$. However for $q > 0$, it takes into account higher representations.

Let's begin by an usual representation of a group $G^{(0)}$. A representation ρ on a finite dimensional vector space is a map.

$$\rho : G^{(0)} \to End(V)$$

where $End(V)$ is the set of endomorphisms of V which is a set of linear maps from V to itself. [9]. In order for it to be a representation, the map ρ needs to satisfy, some following additional conditions. $\rho_g.\rho_{g'} = \rho_{g.g'}$. $\rho_1 = 1$ for all $g.g^{'} \in G^{(0)}$. Finite dimensional vector spaces form a Linear category.

We can denote $q > 0$ representations by $\rho^{(q>0)}$. This not only applies equally to finite , but also to continuous symmetry. So, generalized charges for higher form symmetries can be described as :- q- charges of a higher form symmetry can be denoted as $G^{(q)}_{(q>0)}$

10.2 II. 't hooft anomaly

$d \star J = 0$ holds in the absence of background fields. It gets broken as soon as the symmetry is coupled to a background field and there is no way to rescue this. The partition function which is the path integral that sums over all the topological classes of $U(1)$ bundles can be given as:-

$$\mathcal{Z}[A] \to \mathcal{Z}[A^{\lambda}] = exp(d \star J)\mathcal{Z}[A]$$

This picks up a phase. This can be solved by introducing an anomaly functional , the gauge variation of which will cancel the anomalous variation. The exponentiated action $\mathcal{Z}_{inflow} = e^{-S_{inflow}}$ can be thought of as the partition function of a $5D$ TQFT on X_5, coupled to a background A for a $U(1)$ symmetry . This can be proved with a $3d$ Chern-Simons as well. So far we have done it using $4d$ Maxwell's theory

How to interpret that a global symmetry has a 't Hooft anomaly ?

We can start with a simple $U(1)$ symmetry. It has a global one-form symmetry $\mathbb{Z}$, $U(\mathcal{M}^{(1)}) = exp(i \oint_{M^{(1)}} A)$. The action has to remain invariant under $A \to A + \xi$ (closed 1 - form ξ) , a properly normalized flat $U(1)$ gauge field.[3] The charged objects V are the Wilson lines. These represent $\mathcal{M}^{(1)}$, around a closed manifold $\mathcal{C}^{(1)}$. We can say that the generators of a one-form symmetry are charged under it. We can interpret it to mean that the global symmetry $\mathbb{Z}$ has a 't Hooft anomaly.

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