The need for Gauge Field Theories

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1 Introduction

Let's begin by trying to understand the necessity of studying a field theory approach to quantum mechanics and as the title suggests why we need gauge field theories. We will start off by understanding how gauge theory was formalized, invariance and symmetry and from there we will work our way up to how we can use gauge theory to construct strong and weak interactions (the strong force and the weak force). One thing that needs to be considered while studying field theories , we are trying to understand processes that occur at very small i.e. (quantum-mechanical) scales and very large (relativistic) energies. One might ask why we must study the quantization of fields. Just like the classical formalism approach, why can't we just quantize relativistic particles the way we quantized non relativistic particles? . The answer to that question is simple. When we try to quantize relativistic particles at high energies and shorter distances, the wave equation breaks down. In other words, the Schrodinger equation works for a free particle , but when you try to add relativity to it, it breaks down. We also run into other problems like negative energy solutions, a disorder in first and second order derivatives, negative probability densities etc. Therefore, a totally new framework is needed to deal with particles (fixed number of particles or otherwise) at relativistic levels. This is how quantum field theory was formalized and As we have been hearing since high school, particles are just field excitations acting as operators on a 2D Hilbert space. I am assuming that you as the reader are familiar with concepts like principle of least action, four vectors, Lagrangian and Hamiltonian formalism. If not, check out any MIT OCW playlist.

2 How gauge theory was formalized

If we consider the dynamics of a particle as Sir Issac Newton had formulated in classical mechanics, it was carried into perfection by both Lagrange and Hamilton. The space-time distinction still remained vivid. In field theories, if we consider a field (x, t) the spatial variables x have joined the temporal time variable t. The concept of "Spacetime" has come into being as the

4-dimensional aspect upon which field theories are written. Field theories as a matter of fact have actually have predisposed relativity
for example:- the Hamiltonian formulation of the least action principle:- which suggests that-

$\delta \int_{time-interval} L dt = 0$

The field theory counterpart will be:

$\delta \int_{space-time} \mathcal{L} dxdydzdt = 0$

where $\mathcal{L}$ is the Lagrangian density.
Considering a real scalar field, we can consider its Lagrangian. That is the change in action at a distance and thus derive the Euler Lagrange equations of motion. We can consider several local fields in interaction with one another, giving rise
to complex scalar fields. More on that later.
Now, coming to the question of how gauge theory was formalized , if we consider classical electrodynamics and we try to add relativity to it, we get thrown into certain contradictions. In classical electrodynamics , particles are treated as well defined points, unlike in quantum electrodynamics where there is a combination approach. Considering a point charge, the field produced by a point charge is inversely proportional to the square of the distance from the charge. This as we know is called the coulomb's law. Thus the potential of the field $\phi = \frac{e}{R}$. If we have a system of charges, then the field produced by the system is equal to the superposition of the sum of each field individually.
Given the formula,
R is the distance from the charge e.

$\phi_a = \sum \frac{e_b}{R_{ab}}$

where R_{ab} is the distance between the charges e_a , e_b
But, we know that in relativity , every elementary particle must be considered as point-like. So, at $R->0$, the potential becomes infinity. Thus, according to electrodynamics, the electron would have to have "infinite" self energy and mass. This violates the fundamental framework of classical electrodynamics.
Thus special relativity working conjointly with the principle of superposition throws us into contradictions when it comes to shorter distances.
So, we needed a new "theory of interaction" to design free fields and thus, we came up with gauge theory where the dynamics remain invariant under certain changes/ transformations,

$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \partial_\mu A_\mu$

where A_μ is the gauge field representing interactions and A_μ can be anything.
Let's talk about invariance for a second:-

3 Invariance

Definition:
- Invariance is a broader concept and refers to the property of remaining unchanged under a specific transformation or set of transformations.
- In physics, a system or an equation is said to be invariant under a particular transformation if applying that transformation does not alter the physical laws or properties described by the system or equation.

1. Examples:
- In classical mechanics, Newton's laws are invariant under Galilean transformations. - In special relativity, physical laws are invariant under Lorentz transformations.

2. Mathematical Representation:
- Invariance is often expressed mathematically using transformation rules or equations that remain unchanged after a specified transformation.

4 Lagrangian Invariance:

The Lagrangian formalism in physics is based on the principle of least action, where the dynamics of a system are described by minimizing the action integral. The action (S) of a system is defined as the integral of the Lagrangian $\mathcal{L}$ over time t:

$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt$

where:

$\mathcal{L}$ is the Lagrangian function, which depends on the generalized coordinates q, their time derivatives $\dot{q}$, and possibly time t.
t_1 and t_2 are the initial and final times of the system's motion.
The principle of least action states that the true trajectory of a system between two points in configuration space is the one that minimizes the action integral. The Lagrangian formalism is invariant under certain transformations, such as:

1. Time Translation Invariance:

If the Lagrangian $\mathcal{L}$ does not depend explicitly on time t, i.e., $\frac{\partial \mathcal{L}}{\partial t} = 0$, then the system is invariant under time translations. This implies that the equations of motion derived from the Lagrangian are unchanged if the system's initial time is shifted by a constant.

2. Generalized Coordinate Transformations:

The Lagrangian formalism is invariant under transformations of the generalized coordinates q. If the Lagrangian remains invariant under such transformations, the resulting equations of motion are equivalent. This property is related to the principle of relativity in physics.

3. Symmetry Transformations:

If the Lagrangian remains unchanged under certain symmetry transformations, such as translations, rotations, or gauge transformations, then the resulting equations of motion are invariant under those transformations. This leads to conservation laws, such as conservation of momentum or energy, arising from Noether's theorem.
Mathematically, the invariance of the Lagrangian under a transformation can be expressed as:

$\delta S = \int_{t_1}^{t_2} \delta \mathcal{L} \, dt = 0$

where $\delta \mathcal{L}$ represents the variation of the Lagrangian under the transformation. This condition leads to the Euler-Lagrange equations of motion, which govern the dynamics of the system. Overall, the Lagrangian formalism provides a powerful framework for describing the dynamics of physical systems, and its invariance under various transformations plays a crucial role in understanding the underlying symmetries and conservation laws of nature.
But, since we are interested in field theory, we are going to take a look at lagrangian invariance from a field theory perspective.

5 Invariance for relativistic field equations

From a real scalar field, we can develop its lagrangian density

$L(t) = \int d^3x \mathcal{L}(\phi, \partial_\mu \phi)$

Therefore, the action will be:

$S = \int_{t_2}^{t_1} dt \int \; d^3x \mathcal{L} = \int d^4x \; \mathcal{L}$......eq(1)

Recall that in particle mechanics L depends on q and ˙q.. Similarly, here in field theory it depends on ϕ and $\phi^{\cdot}$
We can determine the equations of motion by the principle of least action. We vary the path, keeping the end points fixed and require δ_S= 0
Considering a real scalar field, we can consider its lagrangian. That is the change in action at a distance and thus derive the euler lagrange equations of motion.
The equations of motion when we expand on eq (1).

$\delta S \ = \ \int \partial^4 x[\frac{\partial \mathcal{L}}{\partial \phi}.\delta\phi \ + \frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi))}.\delta(\partial_\mu \phi)]$

Requiring $\delta_S = 0$, we get the Euler lagrange equations of motions for the path

$\frac{d}{dt}\left(\frac{\partial L}{\partial(\partial_\mu \phi)}\right) - \frac{\partial L}{\partial \phi} = 0$

We can derive our first relativistic field equation from the lagrangian density called the Klein Gordon equation.

From the lagrangian density

$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2$

(You can derive it from $E = mc^2 + p^2c^4$)
we can determine the klein gordon equation by subsituting the value of the lagrangian density in the euler lagrange equation of motion. We are going to get,

$-m^2\phi - \partial_\mu \phi \partial^\mu \phi = 0$

Therefore,

$= \phi(m^2 + \partial_\mu \phi \partial^\mu \phi) = 0$

But there are certain disadvantages like:- negative energy solutions, negative probability densities because it is a second order derivatives and only works for spin 0 particles. Therefore we needed the Dirac equation for the first order derivative, which also works for spin 1/2 particles and has positive probability densities.

$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$

To construct such field theories we need to prove that these theories remain invariant under Lorentz transformation.
But, the Dirac lagrangian density needs to remain invariant, which we will get to in a moment The laws of Nature are relativistic, and one of the main motivations to develop quantum field theory is to reconcile quantum mechanics with special relativity. To this end, we
want to construct field theories in which space and time are placed on an equal footing
and the theory is invariant under Lorentz transformations.

$x \rightarrow (x^{'})^\mu \ = \lambda$

where λ is a function of space time.

The Lorentz transformations have a representation on the fields. The simplest example is the scalar field which, under the Lorentz transformation .

$\phi(x) \rightarrow \phi^{'}(x) = \phi(\lambda^{-1}x)$

The inverse appears in the argument because we are dealing with an active transformation in which the field is truly shifted.

6 Invariance and symmetry

Now, we move on to invariance and symmetry. Invariance as we have known from our high school physics classes, is the when the properties of a system remains unchanged under transformations. In physics, a system or an equation is said to be invariant under a particular transformation if applying that transformation does not alter the physical laws or properties described by the system or equation. A couple of typical examples can be: - In classical mechanics, Newton's laws are invariant under Galilean transformations and in special relativity, physical laws are said to be invariant under Lorentz transformations. Coming to symmetries, in quantum field there are two kinds of symmetry, local symmetry and global symmetry.

A global symmetry is a transformation that acts uniformly on all points in spacetime. Mathematically, a global symmetry transformation is represented by an operator U.
U that commutes with the field operators of the theory. For a scalar field theory, a global symmetry transformation can be expressed as

$\psi(x) \rightarrow \psi'(x) = U\psi(x)\phi(x)$

where $\psi(x)$ is the field operator at spacetime point x, and U is the global symmetry operator. Global symmetries lead to conservation laws through Noether's theorem which states that for every symmetry there is a conserved quantity, whether it be momentum, angular momentum or energy.
For example, a global U(1) symmetry leads to the conservation of electric charge in QED.
A local symmetry, also known as gauge symmetry, is a transformation that varies from point to point in spacetime. In gauge theories, such as Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD), local symmetries are associated with gauge fields and gauge bosons. Mathematically, a local symmetry transformation is represented by a gauge transformation

$U(x)$ that depends on spacetime coordinates

$\psi(x) \rightarrow \psi'(x) = U(x)\psi(x)$

Here, $\psi(x)$ represents the field operator of the fermion field, and U(x) is the gauge transformation. Gauge symmetries introduce redundancy in the description of the theory, which manifests as gauge degrees of freedom. Gauge theories are constructed to be invariant under local gauge transformations, leading to the emergence of gauge fields and ensuring that physical observables are independent of the choice of gauge. More on gauge symmetry, a little later.

7 Mathematical formalism for gauge principle

The gauge principle is a fundamental concept in theoretical physics that states that the laws of physics should be invariant under local transformations of a certain group. In the context of gauge theories, such as electromagnetism and the weak and strong nuclear forces, the gauge principle underlies the symmetries and interactions of elementary particles.

Mathematical formalism:

1. **Gauge Transformations**:

Let's consider a complex scalar field $\psi(x)$ as an example. Under a gauge transformation, the field $\psi(x)$ undergoes a local phase transformation:
$\psi(x) \rightarrow \psi'(x) = e^{i\alpha(x)}\psi(x)$

Here, $\alpha(x)$ is an arbitrary real-valued function of spacetime x.

2.**Gauge Invariance**:

The gauge principle demands that the physical predictions of the theory remain unchanged under such local gauge transformations. Mathematically, this can be expressed as:

$\mathcal{L}(\psi, \partial_\mu \psi, A_\mu) = \mathcal{L}(\psi', \partial_\mu \psi', A_\mu)$

where the gauge field A_μ representing the interaction.

3. **Introduction of Gauge Field**:

To ensure gauge invariance, we introduce a gauge field $A_\mu(x)$ that transforms under gauge transformations such that the gauge-invariant derivative is preserved. This is done by replacing ordinary derivatives with covariant derivatives:

$D_\mu = \partial_\mu - iqA_\mu$

where q is a coupling constant associated with the interaction.

4. **Covariant Derivative**:

Under a gauge transformation, the gauge field A_μ transforms as:

$A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{q}\partial_\mu \alpha(x)$

which can be derived from

$\partial_\mu \psi = \frac{1}{e}(\psi(x + \epsilon.n) - \psi(x))$

(the two fields are subtracted because of different transformations and n^μ is the direction vector)

where ϵ is an infinitesmal change which tends to zero. This can be transformed under an Unitary transformation, which gives us the covariant derivative as,

$D_\mu \psi(x) = d_\mu \psi(x) - iqA_\mu \psi(x)$

We derive the covariant derivative to get the above gauge field A_μ transformation as the above.
The covariant derivative ensures gauge invariance of the Lagrangian.

5. **Gauge Symmetry Group**:

The gauge principle is associated with a gauge symmetry group, such as $U(1)$ for electromagnetism or $SU(2)$ for the weak force. The choice of gauge group depends on the specific theory being considered. The gauge principle is a fundamental concept, underlying the formulation of gauge theories and the understanding of fundamental interactions between elementary particles. It ensures the consistency and invariance of physical laws under local transformations, leading to the introduction of gauge fields and the covariant derivatives that preserve gauge invariance.

8 Gauge symmetry

Now, to ensure gauge invariance, the electromagnetic potential A_μ needs to transform under gauge transformations. It transforms as:

$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$

for a function $\Lambda(x)$ We'll ask only that (x) dies suitably quickly at spatial x i(x= infinity) ! . . We call this a gauge symmetry. The field strength remains locally invariant. We can prove it by taking a Dirac field just like we did in the previous section

$\psi(x) = e^{i\alpha(x)}\psi(x)$ which remains phase invariant under $\alpha(x)$ rotations.

So what are we to make of this? We have a theory with an infinite number of symmetries, one for each function (x). Previously we only encountered symmetries which act the same at all points in spacetime. Noether's theorem told us that these symmetries give rise to conservation laws. Do we now have an infinite number of conservation laws? . The answer is no! Gauge symmetries have a very different interpretation than the global symmetries that we make use of in Noether's theorem. While the latter take a physical state to another physical state with the same properties, the gauge symmetry is to be viewed as a redundancy in our description. That is, two states related by a gauge symmetry are to be identified: they are the same physical state.

9 Gauge Orbit

Since gauge invariance is a redundancy of the system, we might try to formulate the theory purely in terms of the local, physical, gauge invariant objects, both the electric and magnetic field

$\bar{E}$ and $\bar{B}$.

This is fine for the free classical theory: Maxwell's equations
were, after all, first written in terms of $\bar{E}$ and $\bar{B}$.
The picture that emerges for the theory of electromagnetism is of an enlarged phase space, foliated by gauge orbits as shown in the figure. All states that lie along a given
line can be reached by a gauge transformation and are identified. To make progress, we pick a representative from each gauge orbit. It doesn't matter which representative we pick — after all, they're all physically equivalent. But we should make sure that we
pick a "good" gauge, in which we cut the orbits.
Different representative configurations of a physical state are called different gauges.
There are many possibilities, some of which will be more useful in different situations.
Picking a gauge is rather like picking coordinates that are adapted to a particular
problem. Moreover, different gauges often reveal slightly different aspects of a problem. Here we'll look at two different gauges.

10 Gauge fixing

As mentioned earlier, to make progress , we pick a representative from each gauge orbit. It doesn't matter , they are physically equivalent

$A_\mu \rightarrow A_\mu + \partial_\mu \lambda$

where $\lambda\$ is an arbitrary scalar function.$
For any massless vector field, we have to choose a gauge condition to reduce the number of degrees of freedom.
$\partial_\mu A^\mu \lambda = 0$.

This is also known as lorentz gauge. This imposes one constraint. For 4 dimensions , we have 4 degrees of freedom. The gauge condition allows us to eliminate one more degree of freedom.
Therefore, 4(initial components) - 1(Lorentz gauge)= 3 degrees of freedom.
But, this isn't good enough. From Maxwell's equations, we know that $A^\mu = (\phi, \bar{A})$, then the electric field $\bar{E}$ and magnetic field $\bar{B}$ are defined by:

$\bar{E} = -\nabla \phi - \frac{\partial \bar{A}}{\partial t}$ and

$\bar{B} = \nabla X \bar{A}$

and the remaining two are given by the equations of motion

$\nabla . \bar{E} = 0$ and

$\frac{\partial \bar{E}}{\partial t} = \nabla X \bar{B}$.

If we expand , $\nabla . \bar{E} = 0$, we are going to get

$\nabla^2 \phi - \nabla \frac{\partial \bar{A}}{\partial t} = 0$

Now, since we know from Maxwell's equations that $A^\mu = (\phi, \bar{A})$ in that case ,

$\nabla^2 A - \nabla \frac{\partial \bar{A}}{\partial t} = 0$.

We can make use of residual gauge transformation $\nabla . \bar{A} = 0$ (coloumb gauge).
Therefore A = 0.
So, one more degree of freedom has been fixed.

So , 4(initial components) - 1(Lorentz gauge) -1(coloumb gauge) = 2

Hence, only 2 degrees of freedom are needed. So, we have been able to prove that a massless vector field , such as a photon has two degrees of freedom corresponding to its two polarization states. We can also prove it by directly quantizing the electromagnetic field.

11 Prove that the Maxwell Lagrangian is gauge invariant

To prove that the Maxwell Lagrangian is gauge invariant, we need to show that the Lagrangian density

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

remains unchanged under a gauge transformation of the potentials.
1. Gauge Transformation

The gauge transformation of the four-potential A_μ
is given by:

$A'_\mu = A_\mu + \partial_\mu \Lambda$

where Λ is an arbitrary scalar function of space and time.

2. Field Strength Tensor

The field strength tensor $F_{\mu\nu}$ is defined as:

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Under the gauge transformation, the new field strength tensor $F'_{\mu\nu}$ is:

$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$

Substituting the gauge-transformed potentials:

$F'_{\mu\nu} = \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda)$

$F'_{\mu\nu} = \partial_\mu A_\nu + \partial_\mu \partial_\nu \Lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \Lambda$

Since the partial derivatives commute (from commutation relations)

$\partial_\mu \partial_\nu = \partial_\nu \partial_\mu : F'_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$

Thus, the field strength tensor $F_{\mu\nu}$ is invariant under the gauge transformation:

$F'_{\mu\nu} = F_{\mu\nu}$

3. Lagrangian Density

The Lagrangian density for the electromagnetic field is:

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

Under the gauge transformation:

$\mathcal{L}' = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}$

Since we have shown that $F'_{\mu\nu} = F_{\mu\nu}$:
therefore,

$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \mathcal{L}$

So, in conclusion, the Maxwell Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

is indeed gauge invariant because the field strength tensor $F_{\mu\nu}$ remains unchanged under the gauge transformation of the potentials. Therefore, the Lagrangian density itself remains unchanged under gauge transformations, proving the gauge invariance of the Maxwell Lagrangian. The maxwell lagrangian is essential because it describes the propagation of a free fermion.

12 Prove that Dirac Lagrangian density with the covariant derivative is gauge invariant under local gauge transformations.

To apply the gauge principle to the Dirac equation, we start with the Dirac Lagrangian density, which describes the dynamics of a spin-1/2 particle (e.g., electron) in the presence of an electromagnetic field. The Dirac Lagrangian density is given by:

$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

where ψ is the Dirac spinor field, $\bar{\psi}$ is its conjugate, γ^μ are the Dirac matrices, ∂_μ is the partial derivative with respect to spacetime coordinates, and m is the mass of the particle.

We can prove it by taking the Dirac equation and applying the Euler Lagrange equation to it.

Now, let's introduce the gauge principle to this Lagrangian by making it invariant under local gauge transformations. The gauge transformation for the Dirac field ψ is given by:

$\psi(x) \rightarrow \psi'(x) = e^{iq\Lambda(x)}\psi(x)$

where q is the charge of the particle and $\lambda(x)$ is a scalar function of spacetime. To maintain gauge invariance, we need to introduce a covariant derivative D_μ to replace the partial derivative in the Dirac Lagrangian:

$D_\mu\psi = (\partial_\mu - iqA_\mu)\psi$

where, A_μ is the electromagnetic vector potential. Now, let's replace the partial derivatives in the Dirac Lagrangian with covariant derivatives:
To ensure gauge invariance, the electromagnetic potential A_μ needs to transform under gauge transformations. It transforms as:

$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x)$

Substituting this transformation into the expression for the covariant derivative D_μ , we find:

$D_\mu\psi = (\partial_\mu - iqA_\mu)\psi \rightarrow (\partial_\mu - iqA_\mu + iq\partial_\mu\Lambda)\psi$
$= (\partial_\mu - iqA_\mu + iq\partial_\mu\Lambda)\psi$
$= (\partial_\mu - iq(A_\mu + \partial_\mu\Lambda))\psi$
$= (\partial_\mu - iqA'_\mu)\psi'$

Thererfore,

$D'_\mu(x) = \ D_\mu(x)$

Similarly, the Dirac Lagrangian density with the covariant derivative is gauge invariant under local gauge transformations.

$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$
$= \bar{\psi}'(i\gamma^\mu D_\mu - m)\psi'$

Thus, we have successfully applied the gauge principle to the Dirac Lagrangian, ensuring that it remains invariant under local gauge transformations.

13 QED, QED Lagrangian

As we have derived previously the Dirac Lagrangian density, we can see that it acts as interacting theory between an electromagnetic field and spin 1/2 particles. We can formulate the Lagrangian for QED by the quantization of an electromagnetic field , thus giving rise to a photon.
Let's now build an interacting theory of light and matter. We want to write down a Lagrangian which couples the gauge field A to some matter fields, either scalars or spinors. For example, we could write something like: (from The maxwell lagrangian)

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^{\mu}A_{\mu}$

where $j^{\mu}A_{\mu}$ is some function of a matter field.

If we apply the equations of motion, we can get the conserved current.

$d_{\mu}j^{\mu} = 0$

Therefore, the maxwell lagrangian with a conserved current term will be. This term will be useful in a bit.

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial_{\mu}F^{\mu\nu}A_{\mu}$

This term will be useful in a bit.
Now, in order to construct an interaction theory where the scalar field is coupled with a fermion , we first need to prove that current is conserved by Noether's theorem under u(1) symmetry, which is

$j^{\mu}_{\nu} = \bar{\psi}\gamma^{\mu}\psi$

If we recall the Dirac lagrangian density,

$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi$

Under a U(1) symmetry,

$\psi \mapsto e^{i\alpha}\psi$
$\bar{\psi} \mapsto e^{-i\alpha}\bar{\psi},$

we find the Lagrangian is invariant.
Now considering the variation parameter α

$\mathcal{L} \rightarrow \mathcal{L} + \delta\mathcal{L}$ where,

$\delta\mathcal{L} = 0$

As part of Noether's theorem, we find the implicit variation in the Lagrangian due to variation of fields. If the equation of motion for
ψ and $\bar{\psi}$ will be:

$$\delta L = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \delta\bar{\psi} \right)$$

where, $\delta\psi(x) = i\alpha\psi(x)$
This immediately simplifies as there are no partial derivatives of
ϕ and $\bar{\psi}$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu$$
$$0 = \alpha\partial_\mu(\bar{\psi}\gamma^\mu\psi)$$

Therefore, the conserved current will be

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$$

which will be the interaction term when we try to couple an electromagnetic field with fermions
So, For any internal symmetry we can prove that the lagrangian of dirac plus the lagrangian of maxwell plus an interaction remain gauge invariant.

$$\mathcal{L}_{QED-int} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \bar{\psi}\gamma^\mu\psi$$

and this is the Lagrangian of qed with an interaction term.
we, can also prove that the covariant derivative remains gauge invariant

$$D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi \rightarrow e^{-ie\lambda}D_\mu\psi$$

This only picks up a phase λ
This ensures that the whole Lagrangian is invariant, since

$$\bar{\psi} = \rightarrow e^{-ie\lambda(x)}\bar{\psi}$$

If we wish to consider several species of charged particles at once, we simply duplicate $\mathcal{L}$ Dirac and
$\mathcal{L}$ -int for each additional species. And with that we can finally get to understand Feynman rules for scattering amplitudes for photons.

For a scalar particle we can derive the Yukawa lagrangian , It is similar to QED, but with the photon replaced by a scalar particle. The interaction term contains a dimensionless coupling constant g, analogous to the electron charge e. Yukawa originally invented this theory to describe nucleons (psi) and pions (phi). In modern particle theory, the Standard Model contains Yukawa interaction terms coupling the scalar Higgs field (which we will get to) to quarks and leptons; most of the free parameters in the Standard Model are Yukawa coupling constants.

The kind of gauge invariance that usually u(1) symmetry falls under is called Abelian gauge theory, because the gauge is dependent on a single massless vector field.

14 What can gauge theories solve/ the need for gauge invariance

So far in this tutorial, we have been able to prove that the dirac lagrangian as well the maxwell lagrangian remain gauge invariant under gauge transformations, but what can gauge theories actually do for us and why do we need to prove gauge invariance? The reason being that when we interact between fields, a connection develops which is usually the kinetic energy of the gauge. This connection gives rise to a curvature.
The best possible example is probably the electromagnetic field, comprising both electric ($\bar{E}$ and $\bar{B}$) and magnetic field connections. Remember in space time symmetry , when transformations occur along a loop in space time , we call it Riemann curvature. Even if we do not consider space time symmetries, but only think in terms of internal symmetries there is still a curvature .
Think about a quark field, in red, blue and green space, we can transform it around an infinitesimally small space and it will still remain gauge invariant(or the gauge covariant will remain invariant under local gauge transformations).

$q \rightarrow \bar{q}$

The field will be rotated , but the symmetry will be conserved because it remains gauge invariant(Noether's theorem).
For an electron field (internal symmetry) which goes under U(1) transformation:

$\psi_e(x) \longrightarrow^{U(1)} \longrightarrow e^{i\Theta(x)}\psi_e(x)$
The absolute term will remain gauge invariant, producing an electron and a positron.

$|\psi_{e-}|^2 = (\psi_{e-})^*.(\psi_{e-})$

The absolute term will be giving mass to the electron, which corresponds to the mass of the corresponding field. So,

$(\psi_{e-})^*.(\psi_{e-}).\gamma$
will be the interaction term when interacting with a photon. Show the corresponding feynman diagram.

Therefore, the term $|\psi_{e-}|^2$ is gauge invariant.
Now, coupling with matter and coupling with fermions and also when it comes to interactions with different fields, these internal symmetries need to be conserved and thus remain gauge invariant, for example quantum electrodynamics remains gauge invariant under u(1) symmetry and according to the yang mills theory gauge remains invariant under su(n) transformations. There are cases however, when at low energy solutions and lower range the symmetry is not maintained even though the lagrangian remains invariant, this is called spontaneous symmetry breaking and it gives rise to gauge bosons. Unification of different gauge interactions can lead to unified theories. For eg:- The unification of U(1) and SU(2) symmetry leads to the unified electroweak interaction and eventually we can get to a grand unified theory, which can explain all interactions.

15 Constructing strong su(3) interaction and weak interaction with gauge theories

Taking into account what we have discussed earlier, that according to the yang mills theory , gauge remains invariant under SU(n) symmetry transformations. We can simply replace the yang mills lagrangian into the previously discussed maxwell lagrangian and we are going to get the action S for QCD. N= 3 will describe the strong interaction and N =2 will describe the weak interaction. There are $N^2 - 1 = 8$ gauge bosons, which we call gluons, which couple to fermions, which we call quarks, which transform in the defining 3-dimensional representation of SU(3).

The three different values for the index are sometimes labelled by different colours (red, green, and blue), hence the name chromodynamics.

Weak interaction example

If we consider a weak interaction where quarks change their flavour. (su(2) flavour dynamics).
Let's consider an interaction where a neutron gets converted into a proton by exchanging a W^- gauge boson. The W^- is carrying away a negative charge and it bestows a negative charge on the left handed electron upon conservation of charge. The gauge symmetry remains preserved between left handed and right handed electrons from where we choose to see it. A right handed electron gets converted to a left handed electron by the interaction of some field known as the higgs field. when the field $\phi = 0$,
then the potential energy of the marble is also zero. When the marble oscillates from top to bottom, it is at its lowest energy state.

But, the field has some value and can therefore interact with the fields of other particles. The marble can roll around at the bottom of the trough and potential energy is at zero. So no energy is required and that is known as the goldstone boson which is massless because it requires no energy.

The Goldstone theorem states that for every spontaneously broken continuous symmetry, there is a corresponding massless boson

For example, if an SU(N) is spontaneously broken due to an SU(N -1) subgroup, the symmetry group is effectively reduced.

Consider a system with a global

SU(3) symmetry. If this symmetry is spontaneously broken down to due to SU(2), we can count the number of Goldstone bosons as follows:

SU($N^2 - 1$)

Original symmetry = 8(9 - 1)
residual symmetry = 3(4 - 1)
So, therefore broken symmetry will be:- Original - residue = 8 - 3 = 5

SO(5)$\longrightarrow SO(4)$

Original symmetry = 5(5-1)/2 = 10 generators
Residual symmetry = 4(4-1)/2 = 6 generators
Therefore, original - residue = broken.

8 - 3 = 5.

So,
For a SO(5) lie group symmetry there will be 4 goldstone bosons left.
In the case of the electroweak interaction, three of the four would-be Goldstone bosons are "eaten" by the W^+, W^- and Z^0 bosons, giving them mass.

The remaining Goldstone boson corresponds to the fluctuation of the phase of the Higgs field.

Spontaneous symmetry breaking (low energy state) of the SU(2) × U(1) gauge symmetry associated with the electroweak force generates masses for several particles, and separates the electromagnetic and weak forces. The W and Z bosons are the elementary particles that mediate the weak interaction, while the photon mediates the electromagnetic interaction. At energies much greater than 100 GeV, all these particles behave in a similar manner. The Weinberg–Salam theory(see brief history of time) predicts that, at lower energies, this symmetry is broken so that the photon and the massive W and Z bosons emerge.

Now the marble can also oscillate. When it oscillates it is called the higgs boson which essentially gives mass to some of the gauge bosons. Therefore, we can say that spontaneous symmetry breaking is augmented by the Higgs boson to give these particles mass.

The kind of gauge invariance that usually a unified electroweak interaction as well as QCD follows is known as a non abelian gauge theory, as it involves non-linear structure constants like f^{abc}

16 Conclusion

We could on and get deep into N =2 Yang mills theory and brush upon primitively on supersymmetry, but that is for another time. To recap , we started off discussing the lagrangian density and why its invariance is necessary . We get deep into gauge formalism, touch upon u(1) symmetry and qed lagrangian and finally finish off with constructing su(3) and weak interaction. We understand, that we need gauge symmetry and it is realizable at different points, and at these different points, we give rise to new fields and these new fields tell us what is going on at these different points. For different gauge fields , the field strength remains invariant, and from these new connections and different interactions, we can touch upon unified fields.

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