Supersymmetry Formalism from Commutation Relations

Soumyadip Nandi

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1 Introduction

Quantum mechanics is formulated in terms of operators acting on a Hilbert space. The algebra satisfied by these operators determines the kinematics and statistics of quantum systems. Bosonic operators obey commutation relations, while fermionic operators obey anticommutation relations. Supersymmetry arises by extending these two distinct algebraic structures into a single graded algebra capable of transforming bosonic and fermionic degrees of freedom into one another

2 Commutation relations

We are going to start from commutation relations. Now, in order to dictate whether or not two physical observables can be measured simultaneously, we use commutation relations. An important relation in Quantum mechanics is the commutator between two operators. $\hat{A}$ and $\hat{B}$. It is written as $[\hat{A}, \hat{B}]$ and is defined as:-

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

If the commutator is zero, they share common eigenfunctions and are comptabile. If the commutator is non-zero, they follow the Heisenberg's Uncertainty Principle, defining fundamental limits on measurement, polarization etc.

If $[\hat{A}, \hat{B}] = 0$, the two operators are said to commute. $\hat{A}$ and $\hat{B}$ are compatible. $\hat{A}, \hat{B} = \hat{B}\hat{A}$. If the commutator is non-zero, they follow the uncertainty principle.

One of the key implications of commutation relations is **simultaneous measurability**. If $[\hat{A}, \hat{B}] = 0$, the observables A and B can be measured simultaneously with precision. If $[\hat{A}, \hat{B}] \neq 0$, they are incompatible and measuring one disturbs the other. The **Canonical Commutation Relation** is one such instance where the commutation relation fails. Or a better way to phrase it would be:- Bosons are quantized using canonical commutation relations, whereas fermions are quantized using canonical anticommutation relations. This distinction is required by the spin-statistics theorem. There are a few reasons for this. When apllied to finite dimensional Hilbert spaces, the trace of the operator $Tr(AB - BA)$ must be zero. This contradiction results in the failure of CCR. It becomes mathematically invalid. The trace of the identity operator I is non-zero because it represents a sum of eigen values of an identity operator over an infinite dimensional Hilbert space. Broadly speaking, this is the bosonic story.

But first let's discuss the ladder operators.

Considering the **Simple Harmonic Oscillator**, $H = \frac{1}{2}p^2 + \omega^2 p^2$, with the canonical commutation relations $[q, p] = i$ where q_a are the generalized co-ordinates and p^a is the conjugate momenta. We have promoted them to operators. The Poisson bracket structure of classical mechanics morphs into the structure of commutation relations between operators.

$$[q_a, q_b] = [p_a, p_b] = 0$$

$$[q_a, p^b] = i\delta_a^b$$

where , δ_a^b is the **Kronecker Delta Function**. To find the spectrum, we have to define the creation and annihilation operators (also known as ladder operators).

$$a = \sqrt{\frac{\omega}{2}}q + \frac{i}{\sqrt{2\omega}}p, \qquad a^\dagger = \sqrt{\frac{\omega}{2}}q - \frac{i}{\sqrt{2\omega}}p$$

which can easily be inverted to give,

$$q = \frac{i}{\sqrt{2\omega}}(a + q^\dagger), \qquad p = -\frac{i}{\sqrt{2\omega}}(a - q^\dagger)$$

Now, by substituting it to the above equation, we get

$$[a, a^\dagger] = 1$$

While the Hamiltonian is given by:-

$$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \omega(a^\dagger a + \frac{1}{2})$$

These relations ensure that a and $a^\dagger$ takes us between energy eigen states. For bosons, swapping the order of the creation operators ($a^\dagger$) and annihilation operators (a) results in a non-zero commutator. Now, because both the creation and annihilation operators commute with each other, the bosonic states are completely symmetric under the exchange of identical particles.

The canonical commutation relations fails for fermions because their wave function must be anti-symmetric under particle exchange. As mentioned previously, if $[\hat{A}, \hat{B}] \neq 0$, they are incomptabile and measuring one can cause disturbance in the other. it can impose limits like $\Delta x \Delta p \geq \hbar/2$. Non-zero commutator implies that there are just no eigen-states. In classical mechanics, one can assign a value to x and p . So, simultaneous measurability exists at the classical level. When we quantize, we replace the variables with operators.

Now, let's understand why commutation fails entirely for fermions.

Part of the reason for the failure is because there is no commutative classical phase space for fermions. While classical mechanics , models particles as point-like entities with definite trajectories, fermions as mentioned possess intrinsic anti-symmetric wave functions , meaning that they cannot occupy the same quantum state (**Pauli's Exclusion Principle**). The reason why this incompatibility exists is because a "commutative" (traditional) phase-space has no mechanisms to prevent fermions from piling up on the same point in phase-space, which violates the quantum behaviour of fermions.

To put this into a logical format, if we consider the fermionic annihilation operators $\{C, C^\dagger\} = 1$. There is no classical variable $c \in C$ such that $\{C, C\} = 0$. This is because ordinary numbers commute $cc = +cc \neq 0$. So, we can say that for fermions, the obstruction exists even before quantization.

So, a clear description on why bosons have a commutative classical phase space meanwhile the fermions do not can be:-

> ” Bosons admit a commutative classical phase space because their quantum statistics arise only after quantization, whereas fermions do not because their statistics are enforced from the start. For the bosonic system, one can begin with ordinary classical variables $x, p \in \mathbb{R}$ that commute and represent simultaneously well defined configurations; quantization then promotes these variables to operators whose non-commutativity encodes quantum uncertainty , and the classical limit $\hbar \rightarrow 0$ is smooth. For fermions, by contrast are constrained by the Pauli-Exclusion principle and the spin-statistics theorem, which require their fundamental degrees of freedom anti-commute. No set of classical numbers can reproduce this algebra since ordinary numbers cannot square to zero or encode exclusion. As a result, fermions possess no underlying commutative classical phase space at all - their ”classical” description must already be formulated in terms of anti-commutating Grassmann variables , and only bilinear combinations corresspond to physical observables. ”

So, bosons are $[a, a^\dagger] = 1$, fermions = $\{C, C^\dagger\} = 1$, $\{c, c\} = 0$, $C^2 = 0$

Let’s try to understand, how to use Grassmann variables to sove this issue.

3 GrassMann Variables

So, how do we fix this issue ?. Remember that we need to relate the bosonic field ϕ with the fermionic field ψ We arrive at something known as **Grassmann variables**. A grassmann variable, θ is a number like object with one defining feature, $\theta^2 = 0$ and $\theta_1\theta_2 = -\theta_2\theta_1$. $\theta_1^2 = 0$, $\theta_2^2 = 0$ This makes them anti-commute and any power higher than 1 vanishes. This makes Grassmann Variables useful for fermionic degrees of freedom. They encode the Pauli Exclusion principle at the algebraic level.

The appearance of Grassmann variables is not merely a mathematical trick. Their introduction changes the geometry itself.

A couple of points that are essential here:-

- Two identical fermions cannot occupy the same quantum state
- Algebraically squaring a fermionic co-ordinate gives zero. Basically, we have Bosonic $\longrightarrow$ Commuting numbers and fermionic $\longrightarrow$ Anticommuting numbers.

This is a structure that is forced by spin statistics. The reason why path integrals are necessary is because spin 1/2 particles obey Fermi-Dirac statistics with the quantum state picking up a minus sign upon the interchange of any two particles. This fact, as mentioned above is baked into the structure of relavistic QFT. So, to have states that obey fermionic statistics, we need anti-commutation relations. $\{A, B\} = AB + BA$. In this case, the spinor fields should satisfy.

$$\{\psi_\alpha(\overrightarrow{x}), \psi_\beta(\overrightarrow{y})\} = \left\{\psi^\dagger_\alpha(\overrightarrow{x}), \psi^\dagger_\beta(\overrightarrow{y})\right\} = 0$$
$$\left\{\psi_\alpha(\overrightarrow{x}), \psi^\dagger_\beta(\overrightarrow{y})\right\} = \delta_{\alpha\beta}\delta^{(3)}(\overrightarrow{x} - \overrightarrow{y})$$

Now, for commutation relations to work properly, we need the path integrals. To properly represent fermionic fields in a functional integral(sum over histories), the variables must obey the same anti-commutating algebra, thus necessitating the use of Grassmann numbers. The reason why Grassmann numbers are required for anti-commutation to work are as follows.

- In the path integral formulation, integrating over fermionic grassmann variables produces a determinant , $det(D)$, rather than the inverse determinant $(detD)^{-1}$ produced by bosonic fields. This correctly calculates the degrees of freedom for fermions.
- Grassmann Integration rules:- The rules for integration specifically , $\int d\theta = 0$ and $\int \theta d\theta = 1$ might seem bizarre at first , until you realize that the integration actually acts as differentiation, and only linear terms survive.
- Path integrals using Grassmann variables allow for the construction of fermionic coherent states and the correct evaluation of transition amplitudes.

In summary, the path integral for fermions is essentially a formal construction using Grassmann variables to ensure that the "sum over all paths" is anti-symmetric , which is essentially capturing the spin-statistics of fermionic systems. To make the path integral work correctly, we need $\psi(x)\psi(y) = -\psi(x)\psi(y)$.

This is the algebraic foundation for boson-fermion cancellation in SUSY.

Now, we know that we have to relate both the bosonic as well as the fermionic fields for Supersymmetry. The logical path is to start from the Grassmann Harmonic Oscillator (GHO), just like we had done for the Simple Harmonic Oscillator and build our intuition towards SUSY fields.

3.1 GrassMann Harmonic Oscillator

Considering, we have the fermionic creation and annhilation operators:- $\left\{f, f^\dagger\right\} = 1$, $f^2 = (f^\dagger)^2 = 0$. Therefore, the Hamiltonian would be:-

$$H_F = \hbar\omega(f^\dagger f - \frac{1}{2})$$

There are only two states here. The degrees of freedom are intrinsically fermionic. We need to keep in mind that this algebra is not representable by commuting numbers. So, if we want a classical looking formulation (for eg:- a path integral) we must introduce new objects. Its necessary because path integrals manifest symmetry and they can generalize easily to fields and spacetime. As mentioned previously, a Dirac spinor ψ is constructed from two independent grassmann variables $\psi_\gamma(x)$ and $\bar{\psi_\gamma}(x)$, where the components are anti-commuting. $(\psi_i\psi_j - \psi_j\psi_i)$.

So, the grassmann variables $\psi(t)$, $\bar{\psi}(t)$ with $\psi^2 = \bar{\psi}^2 = 0$, $\psi\bar{\psi} = -\bar{\psi}\psi$. The GrassMann Harmonic Oscillator action is:-

$$S = \int dt \;\; \bar{\psi}(i\partial_t - \omega)\psi$$

From here, the equations of motion can be derived. The equations of motion will therefore be:- $(i\partial_t - \omega)\psi = 0$. Hence, the solution :- $\psi(t) = \psi_0 e^{-i\omega t}$.

Here, ψ_0 is Grassmann and we cannot assign a numerical value to it. Only bilinears like $-\bar{\psi}\psi$ are physical.

For the bosonic system, we know that $[a, a^\dagger] = 1$, $H_B = \hbar\omega(a^\dagger a + \frac{1}{2})$.

Now, if we were to consider the combined system, $H = H_B + H_F$, we are going to get,

$$H = \hbar\omega(f^\dagger f + a^\dagger a)$$

If you notice, bosonic = zero-point energy + $\frac{1}{2}\hbar\omega$ and Fermionic = zero-point energy - $\frac{1}{2}\hbar\omega$.

This gets cancelled and is the reason behind why SUSY cancels divergences. Now, we need an operator that can transform fermions into bosons and vice-versa . We can define them as **Supercharges** as fermionic operators.

So, let's define these operators. $Q = af^\dagger$, $Q^\dagger = a^\dagger f$.

These are fermionic operators . For now, all we need to know is that supercharges are described by the Super-Poincare algebra . Supercharge and it's adjoint commute with the Hamiltonian Operator $\{Q, Q^\dagger\} = H$, $Q^2 = (Q^\dagger)^2 = 0$. This is the first glimpse of supersymmetry which is already present in the oscillator. We haven't included spacetime yet, just symmetry.

Note:- The condition $[Q, H] = 0$ means the supercharge Q commutes with the Hamiltonian H. In quantum mechanics, if an operator commutes with the Hamiltonian, it is a conserved quantity (constant of motion). Therefore, the symmetry operator Q maps eigenstates of H to other eigenstates of H with the same energy eigenvalue.

As a conserved quantity, supercharges have another implication:- We know that they transform bosonic states into fermionic states (and vice versa) without changing the energy of the system. This means that for every boson, there is a fermion with the exact same energy, leading to degenerate supermultiplets.

3.2 SUSY Transformations

As per our objective, we need to be able to relate particles with different spin-statistics, specifically mapping bosons to fermions and vice-versa. It is therfore natural to look for an infinitesmal transformation parameter that represents the transformation between fields of opposite GrassMann parity. We can introduce a GrassMann odd infinitesmal parameter ϵ to describe the infinitesmal supersymmetry transformation. $\theta \longrightarrow \theta + \epsilon$, in the fermionic direction.

The grassmann parameters ϵ (Anti-commutating, $\epsilon^2 = 0$) are best known as infinitesmal generators of motion along the fermionic direction in an extended configuration space. So, instead of spacetime alone, x^μ, we can enlarge it into superspace. $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, where x^μ are the Minkowski co-ordinates and θ and $\dot{\theta}$ are the grassmann co-ordinates.

The question now is, why does the parameter become co-ordinates?. In ordinary physics, if a symmetry has a parameter a, and that parameter can vary freely , we can interpret it as a co-ordinate. Now, SUSY gives us a new symmetry with a new parameter ϵ, that behaves consistently like a "direction". So, here we do the same thing , Parameter $\longrightarrow$ Co-ordinate , $\epsilon \longrightarrow \theta$. So, superspace here is the ordinary space as well as the Grassmann directions. Hence, we can now describe systems using $(t, \theta, \bar{\theta})$. This is **superspace** which means that t is how far we are moving in time and θ is how far in the fermionic direction we are moving. Once, we introduce superspace, SUSY transformations become simple translations in θ. Superspace = ordinary space + grassmann.

Bosons and fermions live together in one object (a superfield). A much clearer mental picture here is :- Ordinary space tells you where you are, SUSY tells you what kind of particle you are.

4 Superfields

A superfield is just a function that is defined on the superspace, meaning that it is depends on both ordinary time and Grassmann co-ordinates. So, instead of a field like $x(t)$, we can now consider $\Phi(t, \theta, \bar{\theta})$. Now in Supersymetry Quantum Mechanics , superspace has one-ordinary co-ordinate : t and two Grassmann co-ordinates:- θ and $\bar{\theta}$. They satisfy $\theta^2 = \bar{\theta}^2 = 0$, $\theta\bar{\theta} = -\bar{\theta}\theta$. Where Φ is the chiral superfield.

Because, $\theta^2 = \bar{\theta}^2 = 0$, the most general expansion is:-

$$\Phi(t, \theta, \bar{\theta}) = x(t) + \sqrt{2}\theta\psi(t) + \theta^2 F(t) = 0$$

This is the superfield. Conceptually, we have upgraded from the Fermionic harmonic Oscillator . There's a superpotential after it. The $\sqrt{2}$ is convention. $\psi(t)$ here came as a replacement of f by Grassmann variables. That was just fermionic dynamic. Now, $\psi(t)$ becomes a co-ordinate direction in superspace. The fermionic oscillator is no longer separate , it becomes part of the geometry. $x(t)$ is the bosonic field here. The expansion stops because we cannot have powers that are higher than 1. [Fermionic DOF $\longrightarrow$ co-ordinates].

$\psi(t), \bar{\psi(t)}$ - Fermionic fields. They are Grassmann valued, they anticommute and they encode the exclusion principle. $F(t)$ is the bosonic auxilliary field , and it has no dynamics. There are no kinetic terms for F. If we integrate over the superspace, we can find the action. If we have a single grassmann variable θ , then

$$\int d\theta = 0, \qquad \int d\theta\theta = 1$$

This means that , if we have a function $f(x,\theta) = f_0(x) + \theta f_1(x)$, then the grassmann integration will pick out the component multiplying 0.

$$\int d\theta f(x,\theta) = f_1(x)$$

Here , we need to integrate over the superspace parameterized by θ_α and and $\bar{\theta}^{\dot{\alpha}}$. We define

$$\int d^2\theta = \frac{1}{2}\int d\theta^1\, d\theta^2, \qquad \int d^2\bar{\theta} = -\frac{1}{2}\int d\bar{\theta}^1\, d\bar{\theta}^2.$$

Those strange factors of $\frac{1}{2}$ are because

$$\theta^2 = \theta^\alpha\theta_\alpha = -2\theta^1\theta^2.$$

We then have

$$\int d^2\theta\,\theta^2 = -\int d\theta^1\, d\theta^2\,(\theta^1\theta^2) = 1,$$

where the minus sign disappears when $d\theta^2$ moves past θ^1.

Note that the measure $d^2\bar{\theta}$ comes with an extra minus sign, but this cancels the corresponding minus sign in $\bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = +2\bar{\theta}^1\bar{\theta}^2$.

Once again, we have $\int d^2\bar{\theta}\,\bar{\theta}^2 = 1$.

Finally, we can use the notation

$$\int d^4\theta = \int d^2\theta\, d^2\bar{\theta}.$$

Now, if we were to build an action out of some function of superfields. That function will itself be a superfield that we can call $K(x,\theta,\bar{\theta})$. But K is a composite superfield whose components are functions of other superfields. We can construct the action of the form

$$S = \int d^4x d^4\theta K(x,\theta,\bar{\theta})$$

The action is real if K is a real superfield., obeying $K = K^\dagger$.

So, going back to where we had started from , multiplying a chiral and an anti-chiral superfield together gives us the real superfield. $\Phi^\dagger\Phi$, which we can integrate over the superspace to get the action

$$S_{chiral} = \int d^4x d^4\theta\Phi^\dagger\Phi$$

Now, some calculation and integration by parts shows us that , the action ,

$$S_{chiral} = \int d^4x[\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar{\psi}\sigma^\mu\partial_\mu\psi + F^\dagger F]$$

These are just the standard kinetic terms for a complex scalar ϕ and Weyl fermion ψ. But now we see that there is something special about F: it does not have any kinetic terms. Moreover, this will continue to be true as we write down further supersymmetric interactions. This is what it means to be an auxiliary field. Because there are no kinetic terms for F, it has no propagating degrees of freedom and, when quantised, doesn't give rise to any particle states. The most general renormalizable supersymmetric theory also contains a holomorphic superpotential which has not been derived here.

The remarkable feature of the superfield formalism is that supersymmetry is manifest. Rather than verifying invariance under supersymmetry transformation term by term, one constructs the action directly in superspace. The integral automatically projects out the highest Grassmann component, producing a Lagrangian that is invariant under supersymmetry by construction.

The logical development of supersymmetry can therefore be viewed as a sequence of algebraic structures. One begins with bosonic and fermionic operator algebras, introduces Grassmann variables as the classical analogue of fermionic degrees of freedom, extends spacetime into superspace, constructs supercharges as generators of translations along Grassmann directions, packages fields into superfields, and finally writes a supersymmetric action as an integral over superspace. The entire formalism emerges naturally from the requirement that bosonic and fermionic degrees of freedom be treated within a single graded geometric framework

5 References

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