

The Witten Index

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Supersymmetric quantum mechanics possesses a remarkably rigid structure: the energy spectrum is non-negative, and non-zero energy states come in boson–fermion pairs. This pairing suggests a natural topological invariant — the *Witten index* — which counts the difference between the number of bosonic and fermionic ground states and, crucially, does not change under continuous deformations of the theory. In these notes, we build up to the definition of the Witten index starting from the supersymmetry algebra, explain why it is a robust diagnostic for spontaneous supersymmetry breaking, illustrate it with a concrete example, and finally sketch its path-integral representation.

1. The SUSY Algebra and the Positive-Definite Spectrum

The supercharges Q and $Q^\dagger$ are fermionic operators that generate supersymmetry transformations, and they satisfy the anti-commutation relation

$$\{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q = 2H.$$

Along with the nilpotency conditions $\{Q, Q\} = 0$ and $\{Q^\dagger, Q^\dagger\} = 0$. The anti-commutation relation $\{Q, Q^\dagger\} = 2H$ tells us that the Hamiltonian is the “square” of the supersymmetry generator, ensuring that all energy values are non-negative, $E \geq 0$. Meanwhile, the commutation $[H, Q] = 0$ guarantees that supercharges map energy eigenstates into other states at the same energy.

Furthermore, we see that the energy E is only zero for states $|\psi\rangle$ that are annihilated by both the supercharge and its adjoint. Considering the usual expectation value in a state $|\psi\rangle$:

$$2\langle\psi|H|\psi\rangle = \langle\psi|Q^\dagger Q + QQ^\dagger|\psi\rangle = |Q|\psi\rangle|^2 + |Q^\dagger|\psi\rangle|^2 \geq 0.$$

This follows directly from the anti-commutation relations. For $E = 0$:

$$Q|\psi\rangle = Q^\dagger|\psi\rangle = 0.$$

Already, the idea that we have a positive-definite spectrum is somewhat surprising. Usually in quantum mechanics, we do not really care about the overall energy of states, since

we can always add a constant to the Hamiltonian without changing the physics. But that is not the case for supersymmetric quantum mechanics. The requirement that $E \geq 0$ also rules out some very familiar quantum mechanical potentials, like $V = -1/r$ of the hydrogen atom. The potential in supersymmetric QM must always be positive definite. [Note:- *The positivity of the spectrum should not be confused with positivity of the potential itself. In supersymmetric quantum mechanics the Hamiltonian is positive semi-definite because it is constructed from the supercharges, even though the potential may become negative in some regions of configuration space*]

Considering the set of states with some fixed energy E , we have

$$H|\psi\rangle = E|\psi\rangle.$$

It is simple to check from the supersymmetry algebra that $[H, Q] = [H, Q^\dagger] = 0$, which requires us to also use $Q^2 = Q^{\dagger 2} = 0$. This means that the operators Q and $Q^\dagger$ act within an energy eigenspace. If the energy $E \neq 0$, we have

$$\{Q, Q^\dagger\} = 2E \implies \{c, c^\dagger\} = 1 \quad \text{with} \quad c = \frac{Q}{\sqrt{2E}}.$$

We also have $c^2 = c^{\dagger 2} = 0$. This is the same algebra formed by fermionic creation and annihilation operators. The algebra has a two-dimensional irreducible representation spanned by the states $|0\rangle$ and $|1\rangle$ with the properties $c|0\rangle = 0$ and $c^\dagger|0\rangle = |1\rangle$. The implication of this algebra is that energy states where $E \neq 0$ must come in pairs.

There could be a still bigger degeneracy with several pairs all having the same energy. But at each level, the number of states must be even.

The only exception is $E = 0$. If such a state exists, it is a ground state. It is possible to have a lone state; in that case, any such ground state will obey

$$Q|\psi\rangle = Q^\dagger|\psi\rangle = 0.$$

It is also possible to have more than one ground state. But if that is the case, they are not related by the action of Q and $Q^\dagger$.

2. The Fermion Number Operator and the $\mathbb{Z}_2$ Grading

There is a more formal way of viewing the above story. Inspired by the connection to fermionic creation operators, we can define the “fermion number operator” $F \equiv c^\dagger c$. This obeys

$$[F, Q] = -Q, \quad [F, Q^\dagger] = Q^\dagger, \quad [F, H] = 0.$$

It is important to note that this operator is well defined only on states with energy $E \neq 0$, where it acts as $F|0\rangle = 0$ and $F|1\rangle = |1\rangle$. The $\mathbb{Z}_2$ grading can be extended to the full Hilbert space as a definition, with the convention that zero-energy states are assigned to either the bosonic or fermionic sector depending on whether they are annihilated by Q or $Q^\dagger$ in a way consistent with their transformation properties under rotation.

Correspondingly, the Hilbert space decomposes into “bosonic states” with $F = 0$, and “fermionic states” with $F = 1$:

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F.$$

This is a $\mathbb{Z}_2$ grading of the Hilbert space. The $E \neq 0$ pairs have one state in $\mathcal{H}_B$ and one in $\mathcal{H}_F$. One essential piece of terminology here is that, if a ground state with energy $E = 0$ exists, then we say that supersymmetry is **unbroken**. If the ground state has energy $E > 0$, then we say that supersymmetry is **broken**. This is higher-dimensional language where symmetries that do not leave the vacuum invariant are said to be “spontaneously broken” [3].

So given a theory defined with a Hilbert space $\mathcal{H}$, the main concern is whether there exist theories in $\mathcal{H}$ with zero-energy states. In supersymmetric theories, the energy E is greater than or equal to the magnitude of the momentum $|P|$ for any state. Zero-energy states must therefore have $P = 0$ [5].

In the subspace of states with zero momentum, the supersymmetry algebra is particularly simple. In a basis of properly normalized supersymmetry (super)charges $Q_1, Q_2, \dots, Q_k$ ($k = 4$ for supersymmetry in four dimensions), the algebra is:

$$\begin{aligned} Q_1^2 &= Q_2^2 = \dots = Q_k^2 = H, \\ Q_i Q_j + Q_j Q_i &= 0, \quad \text{for } i \neq j. \end{aligned}$$

These supersymmetry generators map fermions into bosons and bosons into fermions. What is going to be central here, as mentioned previously in connection with the fermion number operator, is $\text{Tr}(-1)^F$ (the parity operator). F here has eigenvalue either 0 or 1 [3]. A bosonic state $|b\rangle$ satisfies $e^{2\pi J_z}|b\rangle = |b\rangle$, and a fermionic state $|f\rangle$ satisfies $e^{2\pi J_z}|f\rangle = -|f\rangle$, where J_z is the infinitesimal rotation generator. A crucial observation here is that the states of non-zero energy E are paired by the action of the supercharges. Let $|b\rangle$ be any bosonic state with non-zero energy E . The action of the supercharges on $|b\rangle$ and $|f\rangle$ is:

$$Q^\dagger|b\rangle = \sqrt{2E}|f\rangle, \quad Q|f\rangle = \sqrt{2E}|b\rangle.$$

This can be checked from the supersymmetry algebra and the creation/annihilation operator formalism above, using $[Q, H] = 0$. All the other states of non-zero energy are paired in two-dimensional supermultiplets with this structure. This gives us a much more rigorous description of why $E \geq 0$. On the other hand, zero-energy states are not paired this way. With $Q^2 = H$, each state annihilated by H is also annihilated by Q . Any bosonic or fermionic state of zero energy satisfies $Q|b\rangle = 0$ or $Q|f\rangle = 0$. They form trivial, one-dimensional supersymmetry multiplets.

3. Parameter Variation and the Witten Index

The question now is: what happens when we vary the parameters of the theory? As we vary the parameters, the states of non-zero energy move around in energy. They move in boson-fermion pairs. So, as the parameters are varied, a state with $E > 0$ may move down to $E = 0$. In this case, $n_B^{E=0}$ and $n_F^{E=0}$ both increase by 1. As the parameters are varied, some states of zero energy may gain non-zero energy. However, it is not possible for a single zero-energy state to acquire a non-zero energy: as soon as it has non-zero energy, it must have a supersymmetric partner. A pair of states can migrate from $E = 0$ to $E \neq 0$. In this case, both $n_B^{E=0}$ and $n_F^{E=0}$ decrease by 1. The quantity $n_B^{E=0} - n_F^{E=0}$ is therefore invariant under parameter variation, and is useful primarily because of two properties:

- (i) It can be calculated reliably.
- (ii) If it is not zero, supersymmetry is not spontaneously broken.

Now, if $n_B^{E=0} - n_F^{E=0} \neq 0$, then obviously $n_B^{E=0} \neq 0$ or $n_F^{E=0} \neq 0$ or both. In any case, there will be some states of zero energy, so supersymmetry is unbroken.

So, what if $n_B^{E=0} - n_F^{E=0} = 0$? In this case, we cannot distinguish between two possibilities:

- (A) $n_B^{E=0} = n_F^{E=0} = 0$: supersymmetry is broken.
- (B) $n_B^{E=0}$ and $n_F^{E=0}$ are equal but non-zero: supersymmetry is unbroken.

If it is (A), then supersymmetry is spontaneously broken, and there is a massless Goldstone fermion. If it is (B), there are no Goldstone fermions, but there are zero-energy fermionic states which can be interpreted as massless fermions. Hence, the quantity $n_B^{E=0} - n_F^{E=0}$ can be regarded as the trace of the operator $(-1)^F$ — this was introduced previously. States of non-zero energy do not contribute to the trace of $(-1)^F$ because for every bosonic state of non-zero energy that contributes +1 to the trace, there is a fermionic state of non-zero energy that contributes -1 and cancels the boson contribution.

Therefore, $\text{Tr}(-1)^F$ can be evaluated among the zero-energy states only, and equals $n_B^{E=0} - n_F^{E=0}$.

We can thus write,

$$\text{Tr}(-1)^F = n_B^{E=0} - n_F^{E=0} \simeq I_W$$

This is the **Witten index**. Now, one could regularize $\text{Tr}(-1)^F$ by writing instead, $\text{Tr}(-1)^F e^{-\beta H}$; for arbitrary positive β . The parameter β plays a role analogous to inverse temperature $\beta = 1/(k_B T)$ in statistical mechanics. The Witten index differs from the usual statistical mechanics partition function by the signs $(-1)^F$.

In supersymmetric theories, the Witten index is actually independent of β :

$$\frac{dI_W}{d\beta} = 0.$$

Why is $dI_W/d\beta = 0$? To see this, expand $\text{Tr}(-1)^F e^{-\beta H}$ in the energy eigenbasis. Each bosonic state of energy $E > 0$ contributes $+e^{-\beta E}$, while its fermionic partner contributes

$-e^{-\beta E}$. These cancel exactly, leaving only the contributions from the $E = 0$ states, which are independent of β . Hence $I_W = n_B^{E=0} - n_F^{E=0}$ does not depend on β at all.

Formally, there is an isomorphism between $\mathcal{H}_B$ and $\mathcal{H}_F$ on the $E \neq 0$ subspace due to the boson–fermion pairing and cancellation. In other words, the Witten index really counts the difference in the number of ground states in each sector.

A comment that is necessary to make is that since I_W does not depend on β , one might wonder why we do not just set $\beta \rightarrow 0$ and consider only the trace, $\text{Tr}(-1)^F$. We need to remember here that $\text{Tr}(-1)^F$ is an infinite series of $+1$ and -1 , and by pairing terms together in various ways, one can obtain any answer one likes. Including $e^{-\beta H}$ in the definition acts as a regulator, rendering the trace finite and unambiguous.

4. A Concrete Realization: SUSY QM on a Line

Now, as we noted at the outset, the Hilbert space decomposes into bosonic and fermionic states: $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$. A $\mathbb{Z}_2$ -graded Hilbert space (often called a super Hilbert space) is a Hilbert space decomposed into even and odd sectors. The anticommutation relation of the supercharges $\{Q, Q^\dagger\} = 2H$, or equivalently $H = \frac{1}{2}\{Q, Q^\dagger\}$, is elegant, but we need to build intuition from a concrete example that realises this algebra [3].

Let us consider the quantum mechanics of a particle moving on a line. The Hilbert space is $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$, where $L^2(\mathbb{R})$ denotes the normalisable functions on the real line $\mathbb{R}$ (the usual Hilbert space for a particle on a line), and $\mathbb{C}^2$ accounts for the internal degree of freedom.

The Hilbert space can then be decomposed into “fermionic” and “bosonic” sectors:

$$\mathcal{H} = L^2(\mathbb{R})|0\rangle \oplus L^2(\mathbb{R})|1\rangle = \mathcal{H}_B \oplus \mathcal{H}_F.$$

In this context, $|0\rangle$ and $|1\rangle$ can be thought of as the spin degree of freedom, with $\mathcal{H}_B$ and $\mathcal{H}_F$ being the “spin down” and “spin up” components of the Hilbert space. For our supercharge Q , we take:

$$Q = (p - ih'(x)) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where $p = -i\partial/\partial x = -i\nabla$ is the momentum operator and $h(x)$ is a real function. We have $Q^2 = 0$ because the 2×2 matrix squares to zero. The conjugate gives us,

$$Q^\dagger = (p + ih'(x)) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

5. The S^1 Example: $I_W = 0$ with Unbroken SUSY

Recall that if $n_B^{E=0} - n_F^{E=0} \neq 0$, SUSY is not spontaneously broken. However, it is not difficult to exhibit examples where $I_W = 0$, yet there exists a pair of bosonic and fermionic $E = 0$ states — so that SUSY is actually unbroken. This is precisely the ambiguity (A) vs.

(B) discussed earlier. A particularly simple example arises from a particle moving on an S^1 circle of radius R .

The supercharge Q and Hamiltonian H take the same form as above and are characterized by a periodic function $h(x) = h(x + 2\pi R)$. We can figure out the family of ground states, and the wave functions can be constructed from them.

The Hamiltonian from the two supercharges was $H = \frac{1}{2} \{Q, Q^\dagger\}$, which gives us:

$$H = \frac{1}{2} \{QQ^\dagger + Q^\dagger Q\} = \frac{1}{2} (p^2 + (h')^2) \mathbb{I} - \frac{1}{2} h'' \sigma_3$$

where $(h')^2$ denotes $(h'(x))^2$. The first factor is the Hamiltonian for a particle with unit mass moving on a line with potential $V(x) = \frac{1}{2}(h')^2$. This term comes with a 2×2 unit matrix $\mathbb{I}$. The second term contains the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Now, for the ground states we require $V(x) = 0 \Leftrightarrow h'(x) = 0$. If we Taylor expand around a critical point $x = x_0$, we have

$$h(x) \simeq h(x_0) + \frac{1}{2} \omega (x - x_0)^2 + \dots$$

While the classical ground state energy of a harmonic oscillator vanishes, quantum mechanically we have $E = \frac{1}{2}|\omega|$. But the supersymmetric system also gets a contribution from the second term in the Hamiltonian, which at leading order is $\Delta E = \pm \frac{1}{2}|\omega|$.

If we take the negative sign, this precisely cancels the contribution from the harmonic oscillator ground state energy, giving us a total semi-classical energy $E = 0$. Now, to determine whether an $E = 0$ ground state exists, consider a general state of the form:

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix}$$

In order to qualify as an $E = 0$ ground state, this must be annihilated by both supercharges: $Q\Psi = Q^\dagger\Psi = 0$. The equations are straightforward: $\psi(x) = e^{-h}$ and $\phi(x) = e^{+h}$.

If $h \rightarrow +\infty$, we must have $\phi(x) = 0$, so the state lives in $\mathcal{H}_B$. If $h \rightarrow -\infty$, we must have $\psi(x) = 0$, so the state lives in $\mathcal{H}_F$. Now, if h has neither of these properties (i.e. h is not bounded in either direction in a way that makes one component normalisable), then there is no $E = 0$ ground state and SUSY is broken. In this case, the ground state energy is non-degenerate (there is a unique ground state at $E > 0$ in each sector, paired by supersymmetry).

Therefore, the wave function we discussed can be constructed from two linearly independent parameters α and β (a two-parameter family of ground states), where $\alpha, \beta \in \mathbb{C}$:

$$\Psi = \alpha \begin{pmatrix} e^{-h} \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ e^{+h} \end{pmatrix}$$

Yet, because one ground state lives in $\mathcal{H}_B$ and the other in $\mathcal{H}_F$, the Witten index of this system is $I_W = 0$. **This is a concrete illustration of case (B) above:** $I_W = 0$, but SUSY is unbroken because there exist both a bosonic and a fermionic zero-energy state. This is precisely the ambiguity that the Witten index alone cannot resolve.

We need to note that understanding whether SUSY is broken or not in a given theory is important for hypothetical phenomenological studies. In fact, this is the main motivation for the Witten index.

6. The Path Integral Representation

From the equation $\text{Tr}(-1)^F e^{-\beta H} \simeq I_W$ for an arbitrary β , we can expand as a functional integral. For four-dimensional theories, the Witten index I_W can be represented as a supersymmetric partition function because, as we have already proven, $E \geq 0$ energy states always exist in pairs. Only the $E = 0$ states contribute, thus making the index independent of continuous parameters like the coupling constant and temperature. While evaluating 4D theories, one can capture this index via path integrals by placing the theory in a specific curved spacetime background [1].

So, to sum it up, the non-zero energy states cancel in pairs under parameter variation. To see how non-zero energy states cancel in the trace, consider any eigenstate $|\psi\rangle$ with $E > 0$. Since $[H, Q] = 0$, the state $Q|\psi\rangle$ (if non-zero) has the same energy E , but the fermion number F changes by one unit because Q is fermionic. Thus, the bosonic and fermionic states at any energy $E > 0$ come in pairs, related by the action of Q , and these pairs contribute with opposite signs in the trace, cancelling exactly. The only states that can survive this pairing are zero-energy states that are annihilated by Q , since $Q|0\rangle = 0$ implies that the state $|0\rangle$ is unpaired. However, as we have seen, boson-fermion pairs can exist at $E = 0$ for specific instances (such as the S^1 example above), and these also cancel in the index.

The Witten index therefore reduces to the difference of the number of unpaired bosonic and fermionic zero-energy states, and more importantly, this difference is an integer which cannot change under any continuous changes in potential, coupling constants, or even the topology of the target manifold in field theory generalizations [1, 2].

Upon expanding the right side of $\text{Tr}(-1)^F e^{-\beta H}$, it can be represented as a functional integral:

$$I_W \propto \int \prod_j dq_j(\tau) \prod_\alpha d\psi_\alpha(\tau) d\bar{\psi}_\alpha(\tau) \exp \left\{ - \int_0^\beta L_E[q_j(\tau), \psi_\alpha(\tau), \bar{\psi}_\alpha(\tau)] d\tau \right\} \quad (1)$$

where L_E is the Euclidean Lagrangian depending on the bosonic (q_j) and fermionic ($\psi_\alpha, \bar{\psi}_\alpha$) dynamic variables in the reduced mechanical system, which satisfies periodic boundary conditions (PBC) in the imaginary time τ :

$$q_j(\beta) = q_j(0); \quad \psi_\alpha(\beta) = \psi_\alpha(0); \quad \bar{\psi}_\alpha(\beta) = \bar{\psi}_\alpha(0).$$

For many systems, the higher Fourier harmonics in the expansion,

$$q_j(\tau) = \sum_n q_j^{(n)} e^{2\pi i n \tau / \beta}, \quad \psi_\alpha(\tau) = \sum_n \psi_\alpha^{(n)} e^{2\pi i n \tau / \beta},$$

can, for small β , be effectively integrated out. The functional integral is then reduced to an ordinary phase-space integral over the constant (zero-mode) field configurations:

$$I_W = \lim_{\beta \rightarrow 0} \int \prod_j \frac{dp_j dq_j}{2\pi} \prod_\alpha d\psi_\alpha d\bar{\psi}_\alpha e^{-\beta H(p_j, q_j; \psi_\alpha, \bar{\psi}_\alpha)} \quad (2)$$

Calculating this integral allows one to evaluate the index in the original theory, or simply [4]:

$$\text{Tr} (-1)^F e^{-\beta H} = \int_{\text{PBC}} d\phi(t) d\psi(t) \exp \{-S_E(\phi, \psi)\} \quad (3)$$

where PBC, as mentioned, denotes periodic boundary conditions with period β , and S_E is the Euclidean action of the theory. Expanding eq. (3) yields eq. (1), and eq. (2) is the reduced functional integral after integrating out the higher Fourier modes.

This path-integral representation is one of the reasons the Witten index is powerful. It provides a bridge between supersymmetric quantum field theory and topology, ultimately leading to results such as the Atiyah–Singer Index Theorem and modern localization techniques.

References

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